Roll No. ....

E–3831

M. Sc./M. A. (Final) EXAMINATION, 2021

MATHEMATICS

(Optional)

Paper Fourth (*ii*)

(Wavelets)

Time : Three Hours] [*Maximum Marks : 100*

Note : Attempt any *two* parts from each Unit. All questions carry equal marks.

Unit—I

1. (a) State and prove Balian-Low theorem for $g \in L^2(R)$.

   (b) Show that the function :

$$\left\{ \frac{W_i}{\sqrt{N}} : j = 0, 1, \ldots.., N-1 \right\}$$

   forms an orthonormal basis for $L_1(Z_N)$.

(c) Let r be a non-negative integer. Let ψ be a function in $C^r(R)$ such that :

$$|\psi(x)| \leq \frac{C}{1+|x|^{r+1+e}}$$

for some $\in > 0$ and that $\psi^{(m)} \in L^\infty(R)$ for m = 1, 2, ..., r. If $\psi_{j,k} |j, k \in Z|$ is an orthonormal system in $L^2(R)$, then all moments of ψ upto order r are zero; i.e. :

$$\int_R x^m \psi(x)\, dx = 0 \quad \forall m = 0, 1, 2, ..., r.$$

Unit—II

2. (a) Suppose N = 1, 2, 3,...., then prove that :

$$\sum_{k \in z} \frac{1}{(\xi + 2k\pi)^{N+1}} = \left(2 \sin\left(\frac{1}{2}\xi\right)\right)^{-N-1} P_N\left(\frac{\xi}{2}\right)$$

where P_N is a trigonometric polynomial satisfying.

(i) P_N is an even function and :

$$P_N(k\pi) = (-1)^{k(N-1)} \quad \forall k \in Z$$

(ii) When N is odd, P_N is π-periodic and :

$$P_N(\xi) > 0 \qquad \forall \xi \in R$$

(iii) When N is even :

$$P_N(\xi + \pi) = -P_N(\xi) \qquad \forall \xi \in R$$

(b) If ψ is band-limited orthonormal wavelet such that $|\hat{\psi}|$ is continuous at 0, then show that $\hat{\psi} = 0$ a.e. in an open neighbourhood of the origin.

(c) If $f \in L^2(T)$, then show that :

$$\{f, Uf, \ldots\ldots, U^N f\}$$

where $U = U_j$ is an orthonormal system if and only if :

$$\sum_{l \in Z} \left| F[f](n + 2^j l) \right|^2 = 2^{-j}$$

for n = 0, 1, 2, ..., N = $2^j - 1$.

Unit—III

3. (a) Prove that :

$$\sum_{j \in D} 2^{-j} \int_{R} \left| \hat{f}(2^{-j}\xi)\,\hat{\psi}(\xi) \right| \sum_{k \neq 0} \left| \hat{f} \;\; 2^{-j}(\xi + 2k\pi) \right.$$

$$\left. \hat{\psi}(\xi + 2k\pi) \right| d\xi < \infty$$

(b) Define low-pass filter. Let $\mu_1, \mu_2, ..., \mu_n$ be 2π-periodic function and set :

$$M_j = \sup_{\xi \in T} \left(\left| \mu_j(\xi) \right|^2 + \left| \mu_j(\xi + \pi) \right|^2 \right)$$

then show that :

$$\int_{-2^n \pi}^{2^n \pi} \prod_{j=1}^{n} \left| \mu_j(2^{-j}\xi) \right|^2 d\xi < 2\pi M_1 .... M_n \; .$$

(c) Prove that ψ is an MRA wavelet iff :

$$D_\psi(\xi) = 1 \text{ a.e. on } \xi \in T$$

and $$g_{m,n}(x) = e^{2\pi i m x} g(x - n)\, m, n \in Z \, .$$

Unit—IV

4. (a) If :

$$\{g_{m,n} : m,n \in \mathrm{Z}\}$$

is a frame for $\mathrm{L}^2(\mathrm{R})$, then show that :

$$\int_{\mathrm{R}} \xi^2 |\hat{g}(\xi)|^2 \, d\xi = \infty .$$

(b) Let $\psi \in \mathrm{L}^2(\mathrm{R})$ be such that :

$$\mathrm{A}_\psi = \underline{\mathrm{S}}_\psi - \sum_{q \in 2\mathrm{Z}+1} \left[\beta_\psi(q)\beta_\psi(-q)\right]^{\frac{1}{2}} > 0$$

and $\mathrm{B}_\psi = \bar{\mathrm{S}}_\psi + \sum_{q \in 2\mathrm{Z}+1} \left[\beta_\psi(q)\beta_\psi(-q)\right]^{\frac{1}{2}} < \infty$

then prove that $\psi_{j,k} \mid j,k \in z$ is a frame with frame bounds A_ψ and B_ψ.

(c) Show that when $\mathrm{Q}_j : j \in \mathrm{J}$ is a frame f can be reconstructed from the coefficients $< f, \mathrm{Q}_j >$ using the dual frame $\mathrm{Q}_j : j \in \mathrm{J}$ and f is also superposition of Q_j, S with coefficients $< f, \tilde{\mathrm{Q}}_j >$.

Unit—V

5. (a) Show that $\tilde{y}_k$ equals $\frac{1}{2}e^{\frac{\pi i k}{2N}}$ times the DCT coefficients $\alpha_k^{(N)}$ for the function *f*.

(b) What do you mean by decomposition of wavelets ? Write in details "how Haar wavelet works for doing the decomposition algorithm."

(c) Show that the sequence :

$$\{u_{j,k} : 1 \le k \le l_j - 1\}$$

is an orthnormal basis for E_j.