Roll No. ....

E–993

M. A./M. Sc. (Fourth Semester) (Main/ATKT)

EXAMINATION, May-June, 2021

MATHEMATICS

Paper Second

(Partial Differential Equations and Mechanics—II)

Time : Three Hours] [*Maximum Marks : 80*

Note : Attempt all Sections as directed.

Section—A 1 each

(Objective/Multiple Choice Questions)

Note : Attempt all questions.

Choose the correct answer :

1. Which is a method of converting non-linear PDE into a linear PDE ?
   (a) Fourier Transform
   (b) Laplace Transform
   (c) Legendre Transform
   (d) Solitons

2. The power series $g = \sum_{\alpha} g_\alpha x^\alpha$ majorizes the power series $f = \sum_{\alpha} f_\alpha x^\alpha$, for all multi-indices α, if :
   (a) $g_\alpha \leq |f_\alpha|$
   (b) $g_\alpha \leq f_\alpha$
   (c) $g_\alpha \geq f_\alpha$
   (d) $g_\alpha \geq |f_\alpha|$

3. For a contact transformation, the correct relation is :
   (a) $\dfrac{\partial q_i}{\partial Q_j} = \dfrac{-\partial P_j}{\partial p_i}$
   (b) $\dfrac{\partial q_i}{\partial Q_j} = \dfrac{-\partial p_i}{\partial P_j}$
   (c) $\dfrac{\partial p_i}{\partial Q_j} = \dfrac{-\partial q_i}{\partial P_j}$
   (d) $\dfrac{\partial p_i}{\partial Q_j} = \dfrac{-\partial P_j}{\partial q_i}$

4. The correct relation between the variations is :
   (a) $\Delta q_r = \dot{q}_r \Delta t + \delta q_r$
   (b) $\delta q_r = \dot{q}_r \Delta t + \Delta q_r$
   (c) $q_{\dot{r}} = \Delta q_r \Delta t + \delta q_r$
   (d) $q_{\dot{r}} = \Delta q_r + \delta q_r \Delta t$

5. Which method does not deal with the analysis of asymptotics ?
   (a) Laplace transform
   (b) Laplace method
   (c) Homogenization
   (d) Stationary phase method

6. In the generating function $F(q_i, p_i, Q_i, P_i, t)$ of the canonical transformation, number of independent variables is :
   (a) $4n+1$
   (b) $4n$
   (c) $2n$
   (d) $2n+1$

7. Which method regards components of gradient of a solution as new independent variables ?
   (a) Holograph Transform
   (b) Legendre Transform
   (c) Laplace Transform
   (d) Fourier Transform

8. The statement "Any other universal integral invariant differs from one of the enumerated integrals by a constant factor" belongs to :
   (a) Hamilton-Jacobi theorem
   (b) Poincare's-integral theorem
   (c) Whittaker's equation
   (d) Lee Hwa-Chung theorem

9. Oscillating solution of wave equation is obtained using :
   (a) Hopf-Cole transform
   (b) Geometric optics
   (c) Plane wave
   (d) Laplace method

10. In Hamilton's characteristic function for a contact transformation :
   (a) new variable are constants of motions
   (b) momenta are constants
   (c) Both (a) and (b)
   (d) None of the above

11. The Cauchy-Kovalevskaya theorem is valid for :
   (a) Linear analytic PDE
   (b) Non-linear analytic PDE
   (c) Both (a) and (b)
   (d) None of the above

12. Characteristic equations for Hamilton-Jacobi PDE are :
   (a) $\dot{X} = D_P H, \dot{p} = D_x H$
   (b) $X = D_P H, \dot{p} = D_x H$
   (c) $\dot{X} = D_P H, \dot{p} = -D_x H$
   (d) $\dot{X} = -D_P H, p = D_x H$

13. 1-D Burger's equation is given by :
   (a) $u_t - au_{xx} + u_x = 0$
   (b) $u_t - a\Delta u + uu_x = 0$
   (c) $u_t - au_{xx} + uu_x = 0$
   (d) $u_{tt} - au_x + uu_{xx} = 0$

14. Hamiltonian 'H' is defined as :
    (a) Total energy of system
    (b) Difference in energy of system
    (c) Product of energy of system
    (d) None of the above

15. A triplet is called admissible if it satisfies :
    (a) Characteristic equations
    (b) Compatibility condition
    (c) Complete integral
    (d) Non-characteristic boundary data

16. The perturbed equation for the transport equation :

$$\operatorname{div}(u\,b) = \delta_0 \text{ in } R^2$$

is :
    (a) $-\in \Delta u^{\in} + div(u^{\in} b) = \delta_0$
    (b) $\in \Delta u^{\in} - div(u^{\in} b) = \delta_0$
    (c) $-\in div(u^{\in}) + \Delta(u^{\in} b) = \delta_0$
    (d) $\in div(u^{\in}) + \Delta(u^{\in} b) = \delta_0$

17. According to Rankine-Hugoniot condition :
    (a) $[[F(u)]] = \sigma + [[u]]$
    (b) $[[u]] = \sigma[[F(u)]]$
    (c) $[[F(u)]] = \sigma[[u]]$
    (d) $[[u]] = \sigma + [[F(u)]]$

18. If (q_i, p_i) be transferred to (Q_i, P_i) by a canonical transformation, then :
    (a) $[Q, P]_{q,p} = [Q, P]_{Q,P}$
    (b) $[Q_i, P_j]_{q,p} = [Q_i, P_j]_{Q,P}$
    (c) $[q, p]_{Q,P} = [Q, P]_{Q,P}$
    (d) $[q, p]_{Q_i, P_j} = [Q, P]_{Q_i, P_j}$

19. Which is a characteristic equation for the PDE $F(Du, u, x) = 0$?
    (a) $\dot{p}(s) = D_x F + p D_z F$
    (b) $\dot{p}(s) = -D_x F + p D_z F$
    (c) $\dot{p}(s) = -D_x F - p D_z F$
    (d) $\dot{p}(s) = D_x F - p D_z F$

20. The solution of Hamilton-Jacobi equation is given by :
    (a) Hopf Lax formula
    (b) Lax-Oleinik formula
    (c) Euler-Lagrange equation
    (d) Local Existence formula

Section—B 2 each

(Very Short Answer Type Questions)

Note : Attempt all questions.

1. State the Local-Existence Theorem.
2. State the Riemann's problem.

3. Define plane and travelling waves.
4. State Cauchy-Kovalevskaya theorem
5. State the principle of least action.
6. State Lee Hwa-Chung theorem.
7. Define Lagrange's bracket.
8. Write the Hamilton-Jacobi equations

Section—C 3 each

(Short Answer Type Questions)

Note : Attempt all questions.

1. Prove that :

$$(u * v)^{\wedge} = (2\pi)^{n/2}\, \hat{u}\, \hat{v},$$

for $u, v \in \alpha'(\mathrm{R}^n) \cap \alpha^2(\mathrm{R}^n)$.

2. Derive the asymptotics in L^∞ norm for the entropy solution given by the Lax-oleinik formula.
3. Show that the transformation :

$$\mathrm{Q} = \frac{1}{p}, \mathrm{P} = qp^2$$

is canonical.

4. Derive Netwon's equation from Hamilton's variational principle.
5. Prove that given properties of Lagrange's bracket :

(i) $\{q_i, p_j\} = \delta_{ij}$

(ii) $\{p_i, p_j\} = 0$

6. Suppose that $k, l : \mathrm{R} \to \mathrm{R}$ are continuous functions that l grows atmost linearly and that k grows at least quadratically. Assume also there exists a unique point $y_0 \in \mathrm{R}$ such that :

$$k(y_0) = \min_{y \in \mathrm{R}} k(y)$$

Then prove that :

$$\lim_{\epsilon \to 0} \frac{\int_{-\infty}^{\infty} l(y)\, e^{-k(y)/\epsilon}\, dy}{\int_{-\infty}^{\infty} e^{-k(y)/\epsilon}\, dy} = l(y_0)$$

7. Solve heat equation using exponential solution.
8. Explain straightening of the boundary.

Section—D 5 each

(Long Answer Type Questions)

Note : Attempt any *four* questions.

1. Find a soliton solution for KdV equation.
2. State and prove Euler-Lagrange equation.
3. Prove that invariance of Lagrange's bracket under canonical transformation.
4. For what values of α and β, do the equations

$$\mathrm{Q} = q^{\alpha} \cos \beta p \text{ and } \mathrm{P} = q^{\alpha} \sin \beta p$$

represent a canonical transformation. What is the form of the generating function F_3 ?

5. Prove that invariance of Poincare Integral with respect to any Hamiltonian system.
6. Explain Homogenization with an example.
7. Derive the convex duality of Hamiltonian and Lagrangian function.
8. Define Fourier transform and find solution of Bessel potential using it.