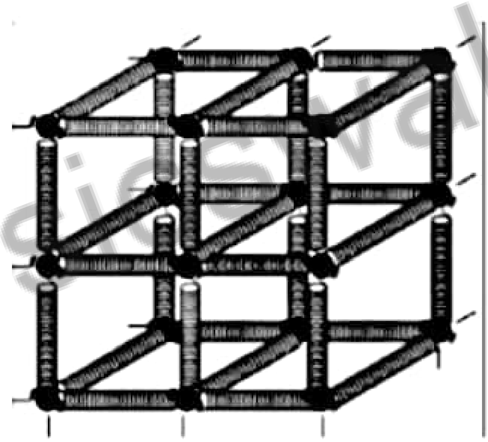
Elasticity

The property of material body by virtue of which its regain its original configuration, when external force is removed is called elasticity.

The property of the material body by virtue of which it does not regain its original configuration when the external force is removed is called plasticity.

Cause of Elasticity

In a solid atoms and molecules are arranged in such a way that each molecule is acted upon by the forces due to the neighbouring molecules. When no deforming force is applied on the body, each molecule of the solid is in its equilibrium position and the inter molecular forces of the solid are maximum. On applying deforming force, the molecules are displaced from their equilibrium position. Inter molecular force gets changed and restoring forces are developed. It is explained by using spring- ball model. Deforming force is removed, these restoring force bring the molecule to its equilibrium positions. Thus the body regains its original shape and size.



Spring-ball model for the illustration of elastic behaviour of solids.

The restoring mechanism can be visualised by taking a model of spring-ball system shown above. Here the balls represent atoms and springs represent interatomic forces.

If you try to displace any ball from its equilibrium position, the spring system tries to restore the ball back to its original position. Thus elastic behaviour of solids can be explained in terms of microscopic nature of the solid. When a body is subjected to a deforming force, a restoring force is developed in the body. This restoring force is equal in magnitude but opposite in direction to the applied force.

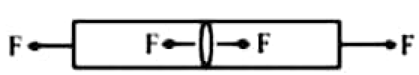
Stress(σ)

When deforming force is applied on the body then the equal restoring force in opposite direction is developed inside the body. The restoring force per unit area is called stress.

$$\text{Stress}(\sigma) = \frac{\text{restoring force}}{\text{Area of cross section of the body}}$$

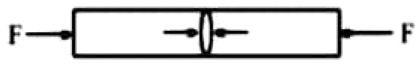
Stress can be tensile or compressive as given below–

Tensile



$\sigma = \frac{F}{A}$ as $F \propto A$

Compressive



Strain

Suppose we stretch a wire by applying tensile forces of magnitude F to each end. The length of the wire increases from L to L + ΔL. The fractional length change is called the strain. It is a dimensionless quantity.

$$\text{strain} = \frac{\Delta L}{L}$$

Hooke's law for tensile and compressive forces

Suppose we had wires of the same composition and length but different thicknesses. It would require larger tensile forces to stretch the thicker wire the same amount as the thinner one. We conclude that the tensile force required is proportional to the cross-sectional area of the wire ($F \propto A$). Thus, the same applied force per unit area produces the same deformation on wires of the same length and composition.

Hooke's Law

$$\text{stress} \propto \text{strain}$$

$$\frac{F}{A} = Y\frac{\Delta L}{L}$$

equation still says that the length change (ΔL) is proportional to the magnitude of the deforming forces (F). Stress and strain account for the effects of length and cross-sectional area ; the proportionality constant Y depends only on the inherent stiffness of the material from which the object is composed ; it is independent of the length and cross-sectional area.

Comparing equation $F = k\Delta L$ and $\frac{F}{A} = Y\frac{\Delta L}{L}$, $F = Y\frac{\Delta L}{L}A$. Y is called the elastic modulus or Young's modulus, Y has the same units as those of stress (Pa or N/m^2) since strain is dimensionless.

Young's modulus can be thought of as the inherent stiffness of a material ; it measures the resistance of the material to elongation or compression. Material that is flexible and stretches easily (for example, rubber) has a low Young's modulus. A stiff material (such as steel) has a high Young's modulus. It takes a larger stress to produce the same strain.

Hooke's law holds up to a maximum stress called the proportional limit. For many materials, Young's modulus has the same value for tension and compression. Some composite materials, such as bone and concrete, have significantly different Young's moduli for tension and compression. The different properties of these two substances lead to different values of Young's modulus for tensile and compressive stress.

Illustration :

A light wire of length 4m is suspended to the ceiling by one of its ends. If its crossectional area is19.6 mm², what is its extension under a load of 10kg. Young's modulus of steel = 2×10^{11} Pa.

Sol. *Given quantities – original length L = 4m ; force F = 10 × 9.8 = 98 N ; and Y = 2×10^{11} Nm^{-2}*

l = ?

Using the relation,

$$\text{Young's modulus (Y)} = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

We have

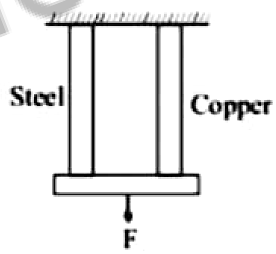
$$Y = \frac{F/A}{l/L} \quad \Rightarrow l = \frac{FL}{YA}$$

$$\therefore \quad l = \frac{98 \times 4}{2 \times 10^{11} \times 19.6 \times 10^{-6}} = 1 \times 10^{-4}\ m$$

$$= 0.1\ mm$$

Illustration :

Two vertical rods of equal lengths, one of steel and the other of copper, are suspended from the ceiling, at a distance l apart and are connected rigidly to a rigid horizontal light bar at their lower ends.



If A_S and A_C be their respective cross sectional areas, and Y_S and Y_C their respective Young's modulii of elasticities, find where should a vertical force F be applied to the horizontal bar, in order that the bar remains horizontal. (Fig.)

Sol. *Let the force F be applied at a distance x from the steel bar, measured along the horizontal bar.*

Let F_S and F_C be the loads on steel and copper rods respectively, so

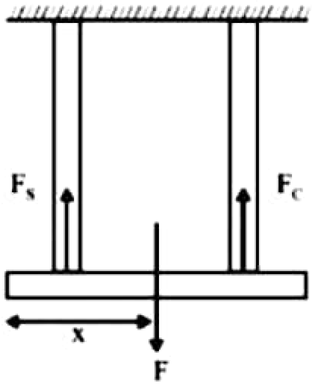
$$F_S + F_C = F \quad \text{... (i)}$$

Since the rigid horizontal bar remains horizontal so, the extensions produced in the two rods and hence strains remains same.

i.e.,

$$\frac{F_S}{A_S Y_S} = \frac{F_C}{A_C Y_C} \quad \text{... (ii)}$$

Solving (i) and (ii) $F_S = \dfrac{FA_SY_S}{A_SY_S + A_CY_C}$



and $F_C = \dfrac{FA_CY_C}{A_SY_S + A_CY_C}$

Now, taking moments about the steel bar.

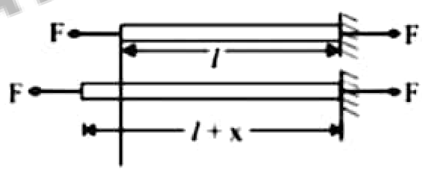
$$F_C l = Fx \quad \Rightarrow x = \frac{F_C}{F} l \;\Rightarrow\; \frac{A_CY_Cl}{A_SY_S + A_CY_C}$$

or $$x = l / \left[1 + \left(\frac{A_S}{A_C}\right)\left(\frac{Y_S}{Y_C}\right) \right]$$

Elastic potential energy

It is the potential energy stored inside the body due to change their configuration. If F force is applied on a body as shown below.

For differential change in length dx the work done by restoring force F is dw



$$dw = -Fdx \qquad \therefore \left(F = \frac{AY}{L}x\right)$$

$$dw = -\frac{AY}{L}x\,dx$$

$$W_{elastic} = -\frac{AY}{L}\int_0^l x\,dx$$

$$\Delta U = -W = \frac{AYl^2}{2L} \quad = \frac{1}{2}\left(\frac{Yl}{L}\right)\left(\frac{l}{L}\right)(AL) \qquad \therefore U_i = 0, U_f = U$$

Elastic potential energy (U) = $\frac{1}{2}$ (stress) (strain) (volume)

Elastic potential energy per unit volume

$$= \frac{1}{2} \times \text{stress} \times \text{strain}$$

The above formula holds good for any type of strain. Change in equilibrium, restoring force = external force F

Then $U = \dfrac{1}{2}\left(\dfrac{YA}{L}\right)l^2$

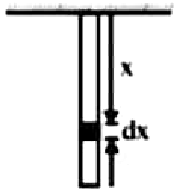
$$= \frac{1}{2}\left(\frac{Y}{L}l\right)Al = \frac{1}{2}Fl$$

Illustration :

A uniform heavy rod of weight W, cross- sectional area A and length L is hanging from a fixed support. Young's modulus of the material of the rod is Y. Neglect the lateral contraction. Find the elongation of the rod.

Sol. *Consider a small length dx of tne rod at a distance x from the fixed end. The part below this small element has* length L – x. The tension T of the rod at the element equals the weight of the rod below it.

$$T = (L - x)\frac{W}{L}$$



Elongation in the element is given by

elongation = original length × stress / Y

$$= \frac{Tdx}{AY} = \frac{(L - x)Wdx}{LAY}$$

The total elongation $= \int_0^L \frac{(L - x)Wdx}{LAY}$

$$= \frac{W}{LAY}\left(Lx - \frac{x^2}{2}\right)_0^L = \frac{WL}{2AY}$$

Illustration:

A wire having a length l = 2m, and cross sectional area A = 5mm² is suspended at one of its ends from a ceiling. What will be its strain energy due to its own weight, if the density and Young's modulus of the material of the wire be d = 9g/cm³ and Y = 1.5×10^{11} Nm^{-2}?

Sol. *Consider an elemental length of the wire of length dx, at a distance x from the lower end. Clearly, this length is acted upon by the external force equal to the weight of the portion of wire below it = xAdg. In equilibrium, the restoring force f = xAdg.*

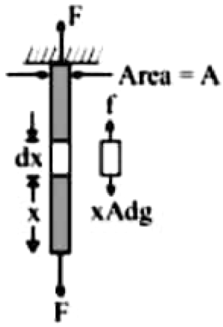
$$\therefore \quad \text{stress} = \frac{f}{A} = xdg.$$

Now, elastic potential energy stored in the elemental length will be

$$dU = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}$$

$$= \frac{1}{2} \times \frac{(\text{stress})^2}{Y} \times \text{volume}$$

$$= \frac{1}{2} \times \frac{(xdg)^2}{Y} \times Adx = \frac{1}{2} \frac{d^2g^2A}{Y} x^2\, dx$$



$\therefore$ *Total elastic potential energy* $U = \int dU$

$$= \int_0^l \frac{1}{2}\frac{d^2g^2A}{Y} x^2\, dx$$

$$= \frac{1}{6} d^2g^2 \frac{Al^3}{Y}$$

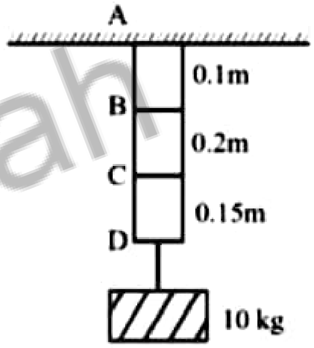
Substituting the values,

$$U = \frac{1}{6} \times \frac{(9\times10^3)(9\cdot8)^2 \times 5 \times 10^{-6} \times 2^3}{1.5 \times 10^{11}}$$

$$= 3.46 \times 10^{-7}\ J$$

Illustration :

Find out the shift in point B, C and D.



$Y_{AB} = 2.5 \times 10^{10}\ N/m^2$

$Y_{BC} = 4 \times 10^{10}\ N/m^2$

$Y_{CD} = 1 \times 10^{10}\ N/m^2$

$A = 10^{-7}\ m^2$

Sol. $$Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L}$$

$$\Rightarrow \quad \Delta L = \frac{FL}{AY} = \frac{MgL}{AY}$$

Shift of point B $(\Delta L_B) = \Delta L_{AB} = \dfrac{10\times10\times0.1}{10^{-7}\times2.5\times10^{10}} = 4 \times 10^{-3}\ m = 4\ mm$

Shift of point C $(\Delta L_C) = \Delta L_B + \Delta L_{BC} = 4 \times 10^{-3} + \dfrac{100\times0.2}{10^{-7}\times4\times10^{10}}$

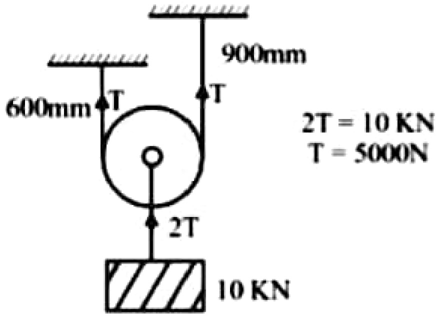
$$= 9 \times 10^{-3}\ m = 9\ mm$$

Shift of point D $(\Delta L_D) = \Delta L_C + \Delta L_{CD} = 9 \times 10^{-3} + \dfrac{100\times0.5}{10^{-7}\times1\times10^{10}}$

$$= 9 \times 10^{-3} + 15 \times 10^{-3} = 24\ mm$$

Illustration :

A load of 10 KN is supported from a pulley which in turn is supported by a rope of cross-sectional area 1×10^3 mm^2 and modulus of elasticity 10^3 N/mm^2, as shown in figure. Neglecting the friction at the pulley determine the deflection of load.



Sol. *longitudinal stress in the rope is*

$$\sigma = \frac{T}{A} = \frac{5 \times 10^3}{10^3 mm^2} = 5 \ N/mm^2$$

$$\text{Extension in the rope} = \frac{\text{stress}}{Y} \times L$$

$$= \frac{5 N/mm^2}{10^3 \ N/mm^2} \times 1500$$

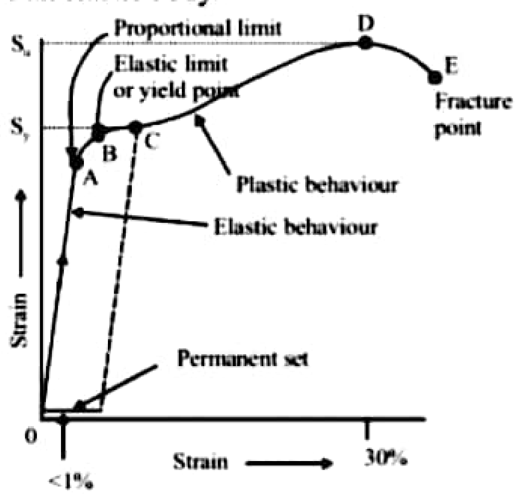
$$= 7.5 \ mm$$

$$\text{Deflection in the load} = \frac{7.5}{2}$$

$$= 3.75 \ mm$$

Stress-strain curve

The relation between the stress and the strain for a given material under tensile stress can be found experimentally. The applied force is gradually increased in steps and the change in length is noted. A graph is plotted between the stress and the strain produced. The stress-strain curves vary from material to material. These curves help us to understand how a given material deforms with increasing loads. From the graph, we can see that in the region between O to A, the curve is linear. In this region, Hooke's law is obeyed. The body regains its original dimensions when the applied force is removed. In this region, the solid behaves as an elastic body.



Stress-strain curve for steel.

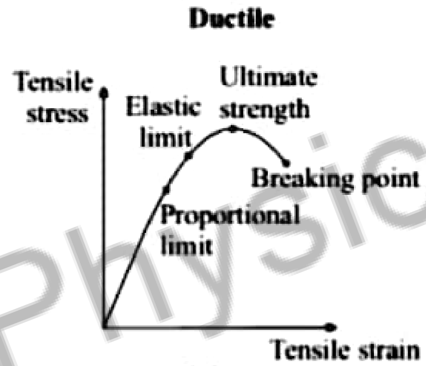
Beyond hooke's law

In the region from A to B, stress and strain are not proportional. Nevertheless, the body still returns to its original dimension when the load is removed. The point B in the curve is known as **yield point** (also known as **elastic limit**) and the corresponding stress is known as **yield strength** (S_y) of the material.

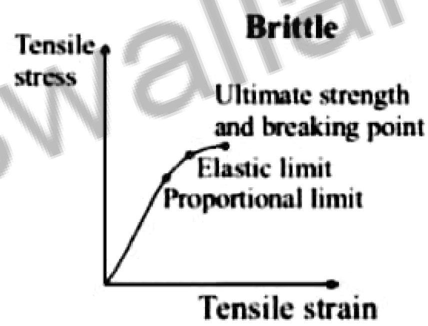
If the tensile or compressive stress exceeds the proportional limit, the strain is no longer proportional to the stress. The solid still returns to its original length when the stress is removed as long as the stress does not exceed the elastic limit.

If the stress exceeds the elastic limit, the material is permanently deformed. For still larger stresses, the solid factures when the stress reaches the breaking point. The maximum stress that can be withstood without breaking is called the ultimate strength. The ultimate strength can be different for compression and tension ; then we refer to the compressive strength or the tensile strength of the material. A ductile material continues to stretch beyond its ultimate tensile strength without breaking ; the stress then decreases from the ultimate strength (fig. (a)). Examples of ductile solids are relatively soft metals, such as gold, silver, copper, and lead. These metals can be pulled like taffy, becoming thinner and thinner until finally reaching the breaking point.

While as Brittle material can not stand beyond ultimate strength



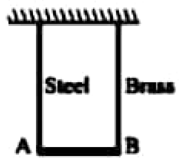
(a)



(b)

Practice Exercise

Q.1 A light rigid bar is suspended horizontally from two vertical wires, one of steel and one of brass, as shown in figure. Each wire is 2.00 m long. The diameter of the steel wire is 0.60 mm and the length of the bar AB is 0.20 m. When a mass of 10 kg is suspended from the centre of AB bar remains horizontal.



(i) What is the tension in each wire?

(ii) Calculate the extension of the steel wire and the energy stored in it.

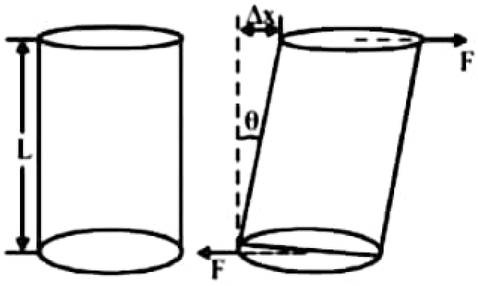
(iii) Calculate the diameter of the brass wire.

(iv) If the brass wire were replaced by another brass wire of diameter 1 mm, where should the mass be suspended so that AB would remain horizontal? The Young modulus for steel = 2.0×10^{11} Pa, the Young modulus for brass = 1.0×10^{11} Pa.

Answers

Q.1 (i) 50 N, (ii) 0.045 J, (iii) 8.4×10^{-4} m, (iv) x = 0.12 m

Shearing Stress



A cylinder subjected to shearing (tangential) stress deforms by an angle θ.

However, if two equal and opposite deforming forces are applied parallel to the cross-sectional area of the cylinder, as shown in fig, there is relative displacement between the opposite faces of the cylinder. The restoring force per unit area developed due to the applied tangential force is known as **tangential** or **shearing stress**.

As a result of applied tangential force, there is a relative displacement Δx between opposite faces of the cylinder as shown in the fig. The strain so produced is known as **shearing strain** and it is defined as the ratio of relative displacement of the faces Δx to the length of the cylinder *L*.

Shearing strain = $\frac{\Delta x}{L} = \tan\theta$

where θ is the angular displacement of the cylinder from the vertical (θ is very samll $\tan\theta \simeq \theta$).

Volume Deformation

Since the fluid presses inward on all sides of the object (figure), the solid is compressed-its volume is reduced. The fluid pressure P is the force per unit surface area ; it can be through of as the volume stress on the solid object. Pressure has the same units as the other kinds of stress: N/m^2 or Pa.

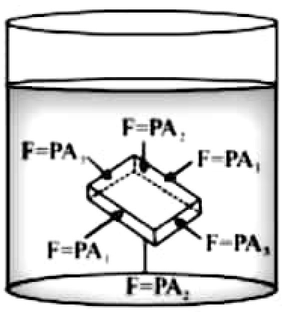


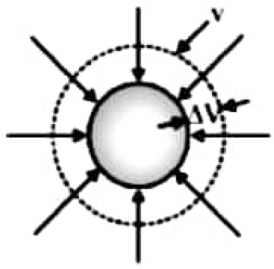
Fig. Forces on an object when submerged in a fluid

$$\text{volume stress} = \text{pressure} = \frac{F}{A} = P$$

The resulting deformation of the object is characterized by the volume strain, which is the fractional change in volume :

$$\text{volume strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{\Delta V}{V}$$

Bulk Modulus (B)



In fig., a solid sphere placed in the fluid under high pressure is compressed uniformly on all sides. The force applied by the fluid acts in perpendicular direction at each point of the surface and the body is said to be under hydraulic compression. This leads to decrease in its volume without any change of its geometrical shape. The body develops internal restoring forces that are equal and opposite to the forces applied by the fluid (the body restores its original shape and size when taken out from the fluid). The internal restoring force per unit area in this case is equal to the hydraulic pressure (applied force per unit area). The strain produced by a hydraulic pressure is called **volume strain** and is defined as the ratio of change in volume (ΔV) to the original volume (V).

$$\text{Volume strain} = \frac{\Delta v}{v}$$

We have seen that when a body is submerged in a fluid, it undergoes a hydraulic stress (equal in magnitude to the hydraulic pressure). This leads to the decrease in the volume of the body thus producing a strain called volume strain.

$$\Delta P = -B\frac{\Delta V}{V} \qquad \text{(Hooke's law for volume deformation)}$$

where V is the volume at atmospheric pressure. The negative sign. equation $\Delta P = -B\frac{\Delta V}{V}$ allows the bulk modulus to be positive. The bulk moduli of liquids are generally not much less than those of solids, since the atoms in liquids are nearly as close together as those in solids.

Gases are much easier to compress than solids or liquids, so their bulk moduli are much smaller. The bulk moduli of a few common materials are given in Table

Material	B (10^9 Nm^{-2} or GPa)
Solids	
Aluminium	72
Brass	61
Copper	140
Glass	37
Iron	100
Nickel	260
Steel	160
Liquids	
Water	2.2
Ethanol	0.9
Carbon disulphide	1.56
Glycerine	4.76
Mercury	25
Gases	
Air (at STP)	1.0×10^{-4}

Table : Bulk moduli *(B)* of some common Materials

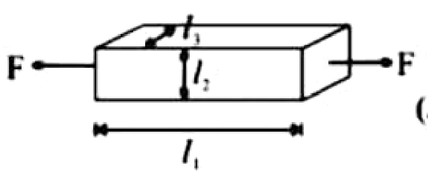
Compressibility (k)

The reciprocal of the bulk modulus is called compressibility and is denoted by k. It is defined as the fractional change in volume per unit increase in pressure.

$$k = \frac{1}{B} = -\frac{1}{V}\left(\frac{\Delta V}{\Delta P}\right)$$

Poisson's ratio

When an elongation is produced by longitudinal stresses, a change is produced in the lateral dimensions of the strained substance. Thus, when a wire is stretched, its diameter diminishes ; and when the longitudinal strain is small, the lateral strain is proportional to it. The ratio of the lateral strain to the longitudinal strain is called Poisson's ratio.



(l_1, l_2, and l_3 are the dimentional when no strain. Δl_1, Δl_2, and Δl_3 are the change in length of l_1, l_2, and l_3 respectively)

$$Y = \frac{\frac{F}{A}}{\frac{\Delta l_1}{l_1}}$$

$$\frac{\Delta l_2}{l_2} = \frac{\Delta l_3}{l_3} = -\sigma\frac{\Delta l_1}{l_1}$$

Illustration:

. *A uniform bar of length L and cross sectional area A is subjected to a tensile load F. If Y be the Young's modulus of the material of the bar and σ be its poisson's ratio, then determine the volumertic strain.*

Sol. *Longitudinal stress* $= \frac{F}{A}$.

Longitudinal strain $= \frac{F}{AY} = \varepsilon_l$ *(say)* ... (i)

Now, by definition of Poisson's ratio,

$$\sigma = \frac{-\text{lateral strain}}{\text{longitudinal strain}} = \frac{-\delta r / r}{\delta L / L}$$

or $\delta r / r = -\sigma \delta L / L \Rightarrow -\frac{\sigma F}{AY}$ *[From eqn. (i)]*

Since Volumetric strain = Strain in length + Twice strain in radius.

$\therefore$ *Volumetric strain* $= \frac{\delta L}{L} + \frac{2\delta\delta}{r}$

$$= \frac{F}{AY} + 2\left(-\frac{\sigma F}{AY}\right) = \frac{F}{AY}(1 - 2\sigma).$$