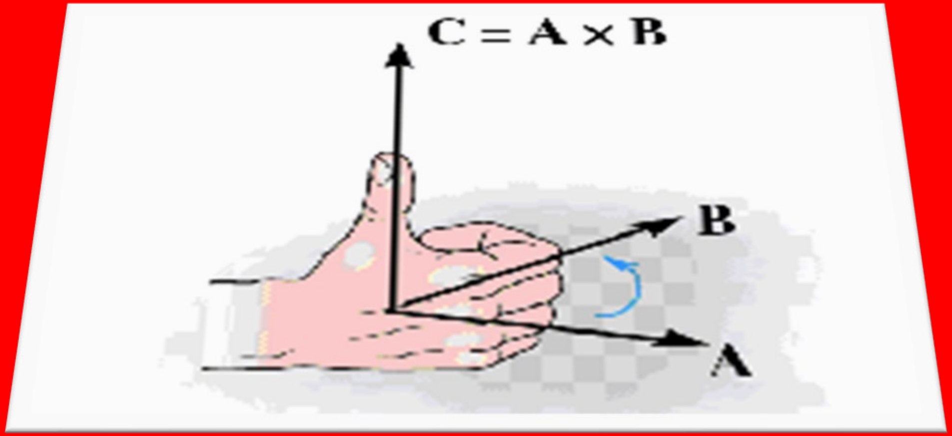
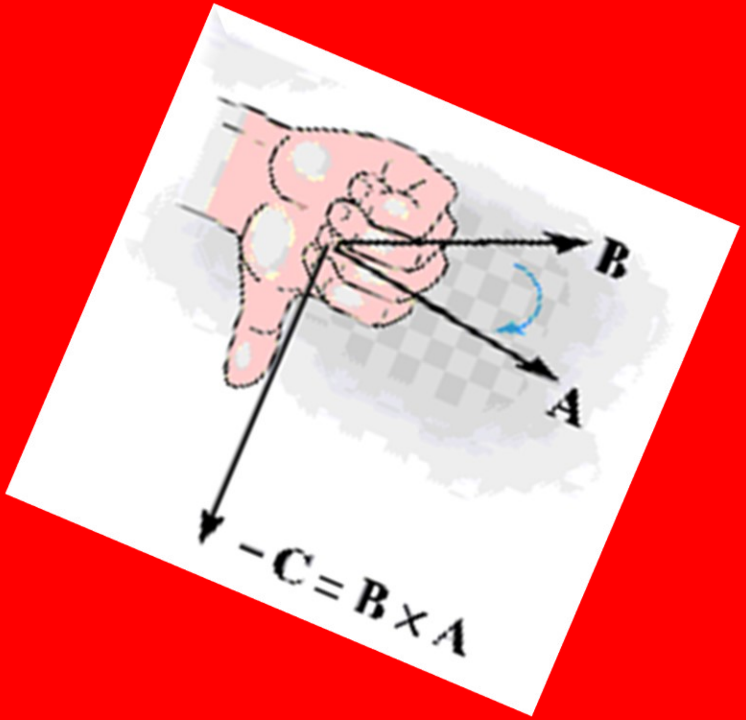
Alpha Physics

Ch-04 Vectors

Lect-11





Today's Goal

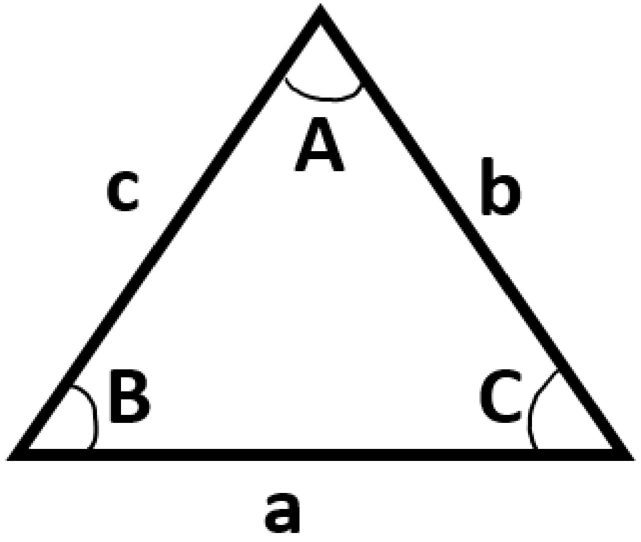
1. Sine Law and Cosine Law
2. Direction Cosines
3. Volume of Parallelepiped
4. How to Check if 3 vectors are Coplanar

1. Sine Law

Prove using Vector Algebra

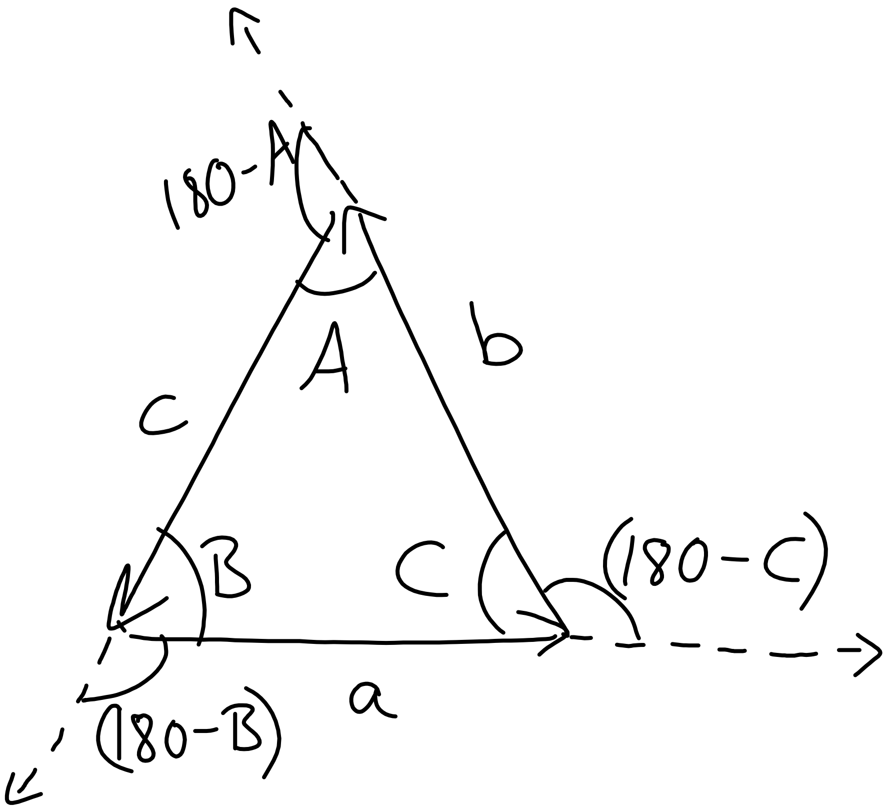
$$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$$

True for all Triangles



Sides $\longrightarrow$ a, b, c

To Prove $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$



$$\boxed{\vec{a} + \vec{b} + \vec{c} = 0}$$

{ Polygon Law
Head-Tail Method }

$$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times 0$$

$$\underbrace{\vec{a} \times \vec{a}}_{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{a} \times 0$$

$$\vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$$

$$\vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$$

$$\vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$$

$$|\vec{a} \times \vec{b}| = |-\vec{a} \times \vec{c}|$$

$$|\vec{a} \times \vec{b}| = |\vec{a} \times \vec{c}|$$

$$\cancel{a}\, b \sin(180 - C) = \cancel{a}\, c \sin(180 - B)$$

$$b \sin C = c \sin B$$

$$\frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{—— (i)}$$

$$|-\vec{x}| = |\vec{x}|$$

$$\sin(180 - \theta) = \sin\theta$$

$$\vec{a} + \vec{b} + \vec{c} = 0$$

$$\vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times 0$$

$$\vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = \vec{b} \times 0$$

$$\vec{b} \times \vec{a} + 0 + \vec{b} \times \vec{c} = 0$$

$$|\vec{b} \times \vec{a}| = |-\vec{b} \times \vec{c}|$$

$$|\vec{b} \times \vec{a}| = |\vec{b} \times \vec{c}|$$

$$|-\vec{X}| = |\vec{X}|$$

$$ab\sin(180-C) = bc\sin(180-A)$$

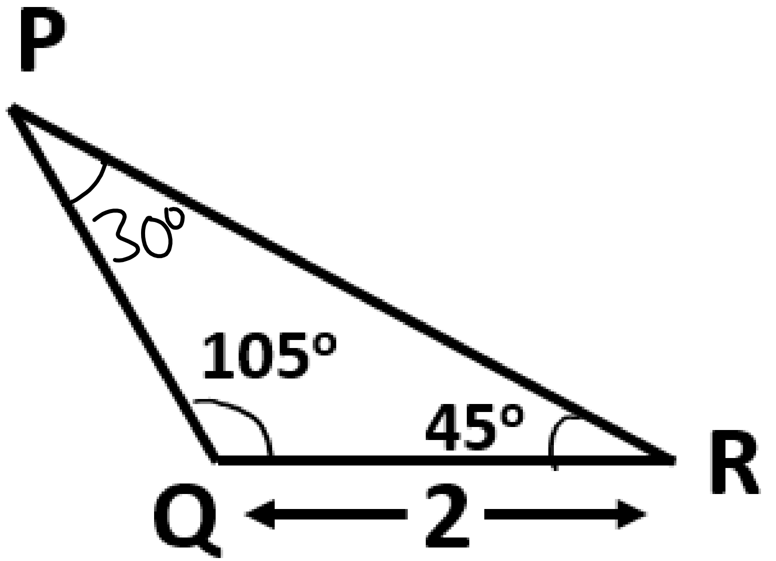
$$a\cancel{b}\sin C = \cancel{b}c\sin A$$

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \text{---(ii)}$$

From (i) & (ii)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Q) Find PQ



Sine Law

$$\frac{PQ}{\sin 45^\circ} = \frac{QR}{\sin 30^\circ} = \frac{PR}{\sin 105^\circ}$$

$$\frac{PQ}{\sin 45^\circ} = \frac{QR}{\sin 30^\circ}$$

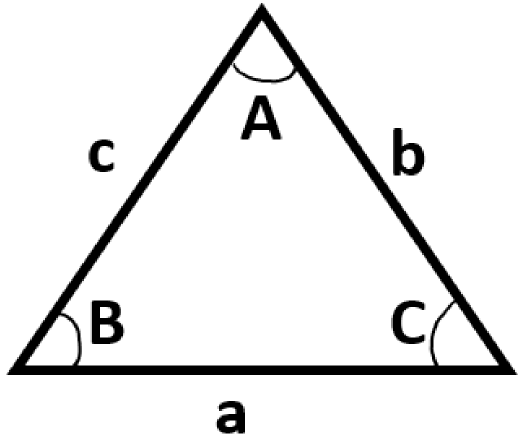
$$\frac{PQ}{\frac{1}{\sqrt{2}}} = \frac{2}{\frac{1}{2}}$$

$$PQ = 4 \times \frac{1}{\sqrt{2}} = \frac{2 \times 2}{\sqrt{2}}$$

$$= \frac{2 \times \sqrt{2}\,\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} = 2\sqrt{2}$$

2. Cosine Law

Prove using Vector Algebra



True for any Triangle

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{b^2 + a^2 - c^2}{2ab}$$

To Prove

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

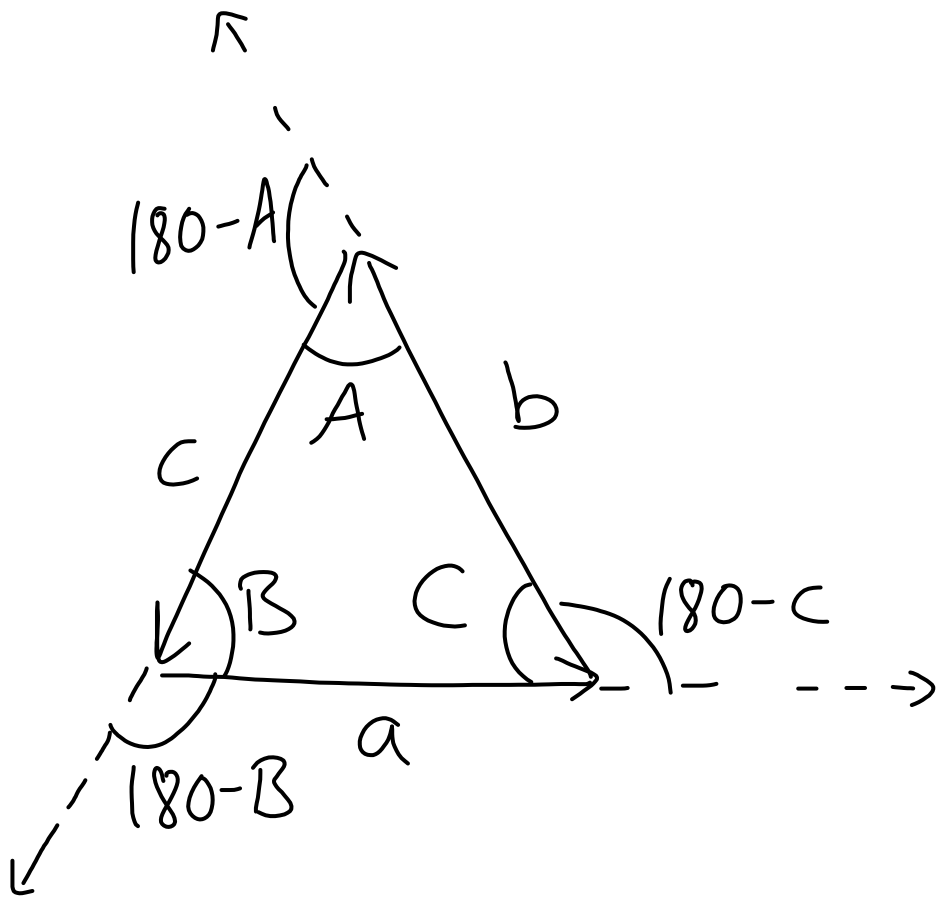
$$\vec{a}+\vec{b}+\vec{c}=0$$

(Polygon law)

$$\vec{b}+\vec{c}=-\vec{a}$$

$$|\vec{b}+\vec{c}|=|-\vec{a}|$$

$$|\vec{b}+\vec{c}|^2=|\vec{a}|^2$$



$$|\vec{b}+\vec{c}|^2 = |\vec{a}|^2$$

$$\cos(180-\theta) = -\cos\theta$$

$$\sin(180-\theta) = \sin\theta$$

$$\left(\sqrt{b^2+c^2+2bc\cos(180-A)}\right)^2 = |\vec{a}|^2$$

$$\left(\sqrt{b^2+c^2-2bc\cos A}\right)^2 = a^2$$

$$b^2+c^2-2bc\cos A = a^2$$

$$2bc\cos A = b^2+c^2-a^2$$

$$\boxed{\cos A = \frac{b^2+c^2-a^2}{2bc}}$$

To Prove

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\vec{a}+\vec{b}+\vec{c}=0$$

$$\vec{a}+\vec{c}=-\vec{b}$$

$$|\vec{a}+\vec{c}|=|-\vec{b}|$$

$$|\vec{a}+\vec{c}|^2=|\vec{b}|^2$$

$$\left(\sqrt{a^2+c^2+2ac\cos(180-B)}\right)^2 = b^2$$

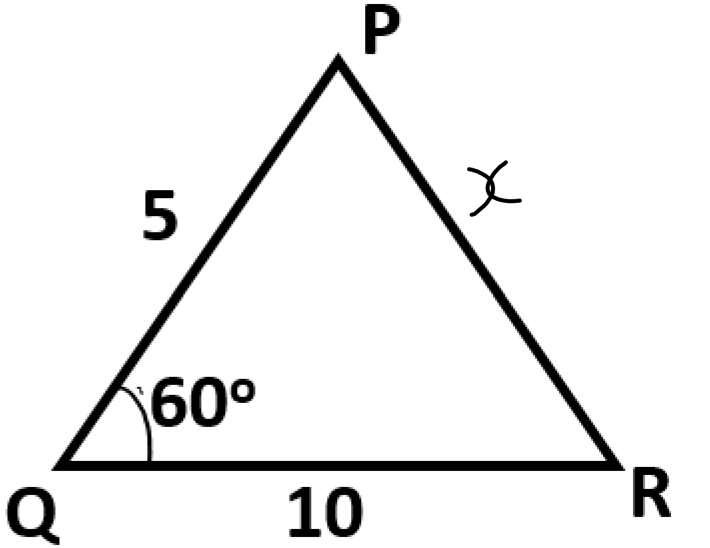
$$a^2+c^2-2ac\cos B = b^2$$

$$2ac\cos B = a^2+c^2-b^2$$

$$\boxed{\cos B = \frac{a^2+c^2-b^2}{2ac}}$$

Try for Cos C

Q) Find PR



Cosine law

$$\cos 60^\circ = \frac{10^2 + 5^2 - x^2}{2 \times 5 \times 10}$$

$$\frac{1}{\cancel{2}} = \frac{100 + 25 - x^2}{\cancel{2} \times 5 \times 10}$$

$$50 = 125 - x^2$$

$$x^2 = 125 - 50 = 75$$

$$x = \sqrt{75} = \sqrt{25 \times 3} = 5\sqrt{3}$$

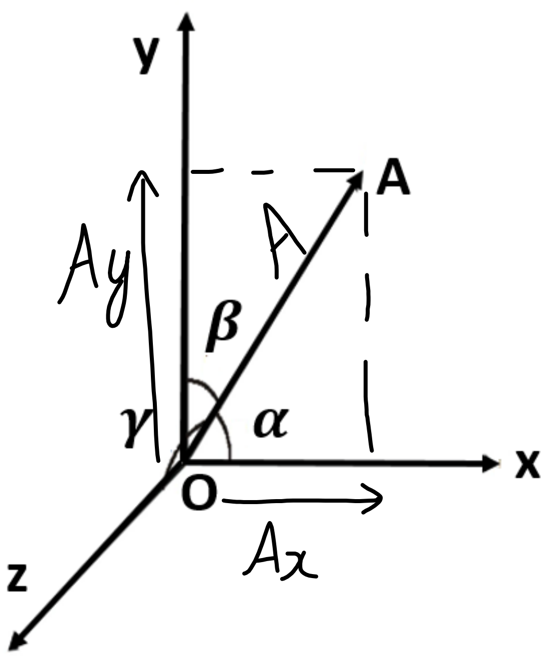
$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Direction Cosines (l, m, n)

$l \to x$, $m \to y$, $n \to z$

$$l = \cos\alpha\,;\, m = \cos\beta\,;\, n = \cos\gamma$$

Prove that $l^2 + m^2 + n^2 = 1$



Property of direction cosines

$$\cos\alpha = \frac{A_x}{A}, \quad \cos\beta = \frac{A_y}{A}$$

$$\cos\gamma = \frac{A_z}{A}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{A_x}{A}\right)^2 + \left(\frac{A_y}{A}\right)^2 + \left(\frac{A_z}{A}\right)^2$$

$$(l)^2 + (m)^2 + (n)^2 = \frac{A_x^2}{A^2} + \frac{A_y^2}{A^2} + \frac{A_z^2}{A^2}$$

$$l^2 + m^2 + n^2 = \frac{A_x^2 + A_y^2 + A_z^2}{A^2}$$

$$l^2 + m^2 + n^2 = \frac{A^2}{A^2}$$

$$l^2 + m^2 + n^2 = 1$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$|A|^2 = A_x^2 + A_y^2 + A_z^2$$

$$A^2 = A_x^2 + A_y^2 + A_z^2$$

Q1) A vector P makes angle α, β, γ with X,Y and Z axis respectively. Find the value of

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma =$$

a) 1

b) 2 ✓

c) -1

d) -2

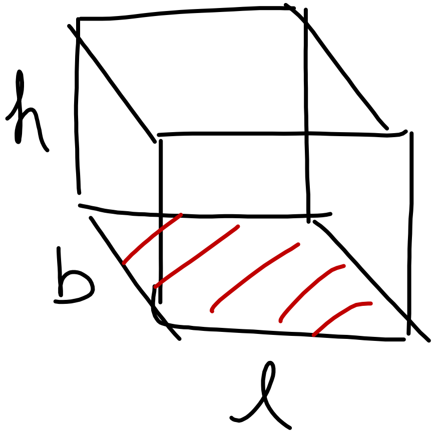
$\alpha, \beta, \gamma \rightarrow$ with X, Y, Z axis

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$

$$2 = \sin^2\alpha + \sin^2\beta + \sin^2\gamma$$

Volume of a Parallelepiped

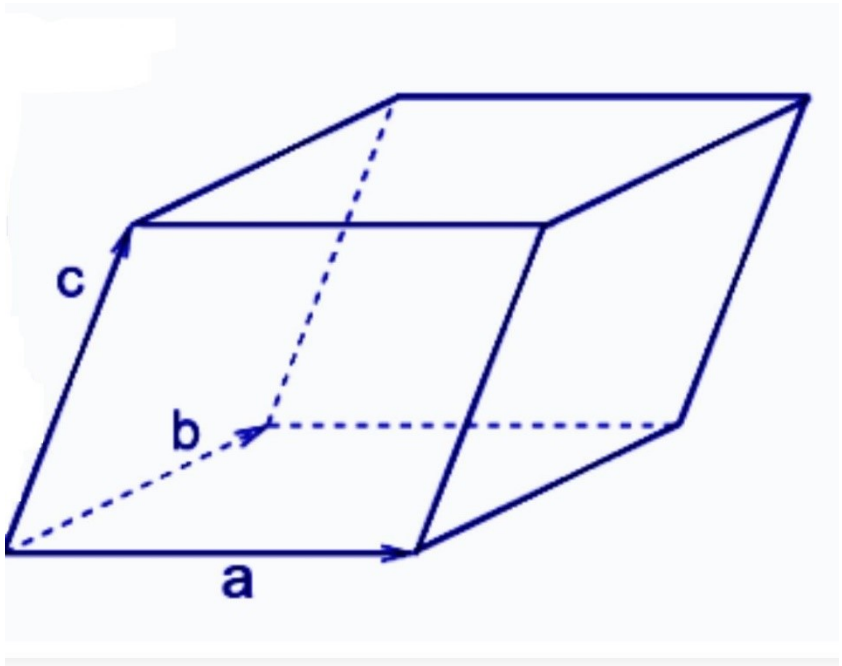


$$\text{Volume} = l\, b\, h$$

$$\text{Volume} = \text{Area of base} \times \text{height}$$

$$\text{Volume} = \pi R^2 h$$

$$\text{Volume} = \text{Area of base} \times \text{height}$$



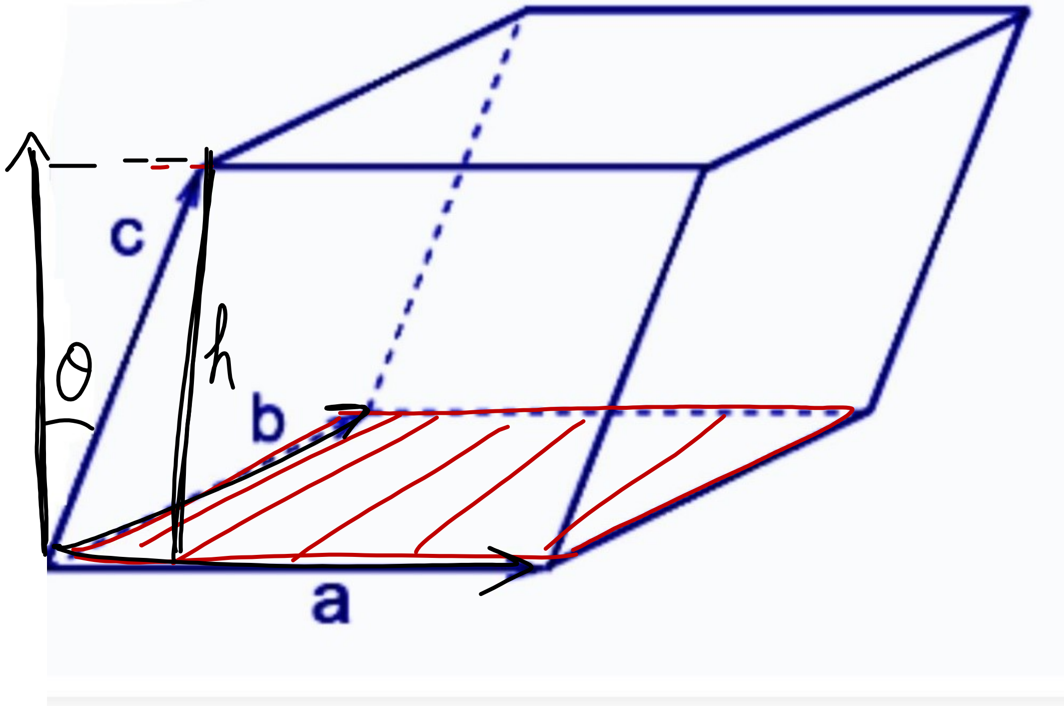
$h = c\cos\theta$

$$\text{Volume} = \text{Area of base} \times \text{height}$$

$$= \text{Area of Parallelogram} \times \text{height}$$

$$= |\vec{a} \times \vec{b}| \times \text{height}$$

$$= (\vec{a} \times \vec{b})(c\cos\theta)$$



Next Page

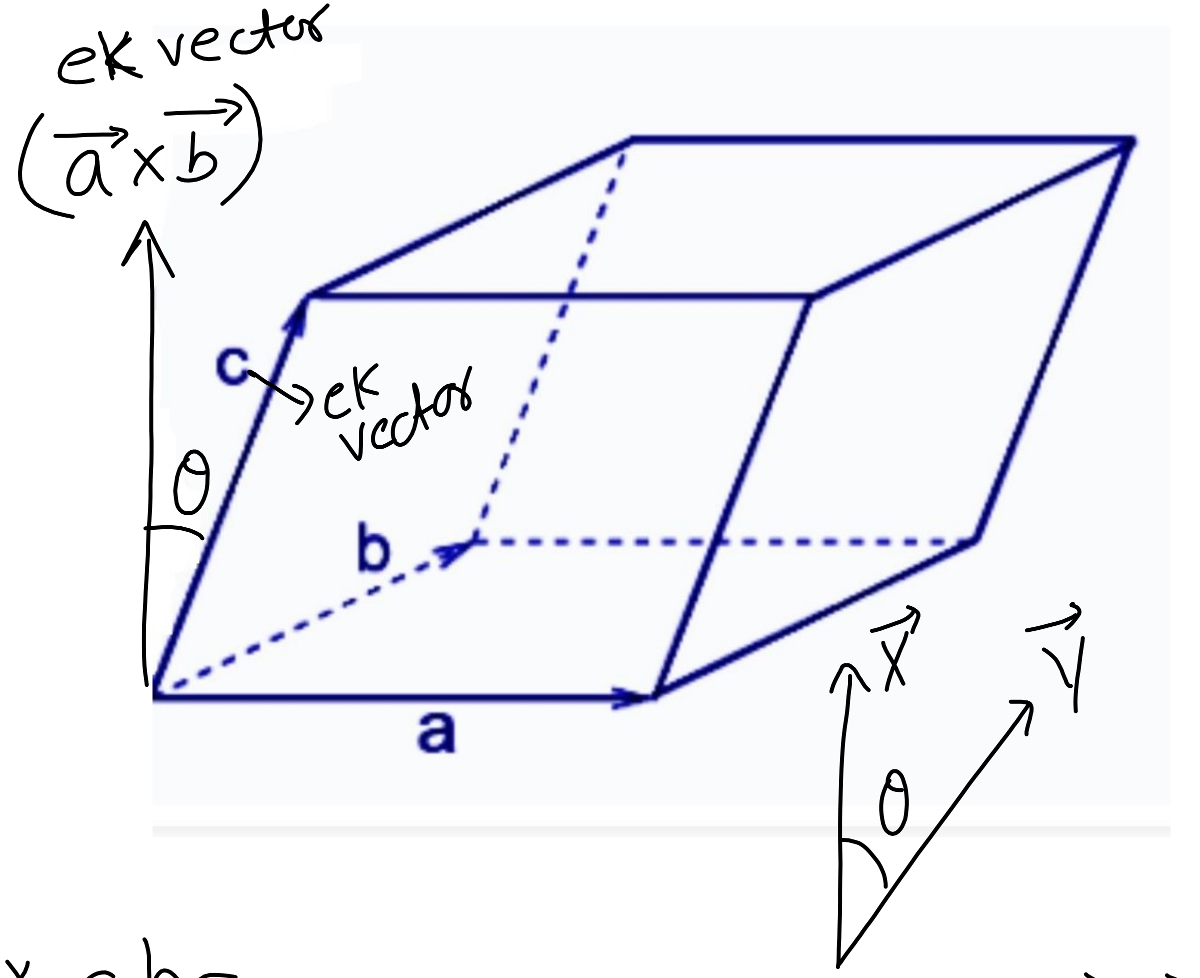
$$\text{Volume} = \underbrace{(\vec{a}\times\vec{b})}_{\text{ek vector}}\,(\vec{C}\cos\theta)$$

$\vec{C} \rightarrow$ ek vector

$$\boxed{\text{Volume} = (\vec{a}\times\vec{b})\cdot\vec{C}}$$

Scalar Triple Product

$$(\vec{a}\times\vec{b})\cdot\vec{C} = [a\,b\,c] = \text{Box } abc$$



$$X\,Y\cos\theta = \vec{X}\cdot\vec{Y}$$

Volume of a Parallelopiped $= \left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right|$

$$= \left| [a\ b\ c] \right|$$

Note:

$$[a\ b\ c] = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{c} \cdot (\vec{a} \times \vec{b})$$

$$= (\vec{b} \times \vec{c}) \cdot \vec{a} = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$= (\vec{c} \times \vec{a}) \cdot \vec{b} = \vec{b} \cdot (\vec{c} \times \vec{a})$$

Q2) Find the volume of parallelepiped (in m^3) whose edges are represented by

$\vec{a}=(2\hat{\imath}-3\hat{\jmath}+4\hat{k})m$ $\vec{b}=(\hat{\imath}+2\hat{\jmath}-\hat{k})m$ $\vec{c}=(3\hat{\imath}-\hat{\jmath}+2\hat{k})m$

a) 6

b) ~~7~~

c) 8

d) 9

$$\text{Volume} = \left|[a\,b\,c]\right|$$

$$= (\vec{a}\times\vec{b})\cdot\vec{c}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 1 & 2 & -1 \end{vmatrix}$$

$$\hat{i}(3-8) - \hat{j}(-2-4) + \hat{k}(4+3)$$

$$= (-5\hat{i}+6\hat{j}+7\hat{k})\cdot(3\hat{i}-\hat{j}+2\hat{k})$$

$$= -15-6+14$$

$$= -21+14 = |-7| = 7$$

How to Check if 3 vectors are Coplanar

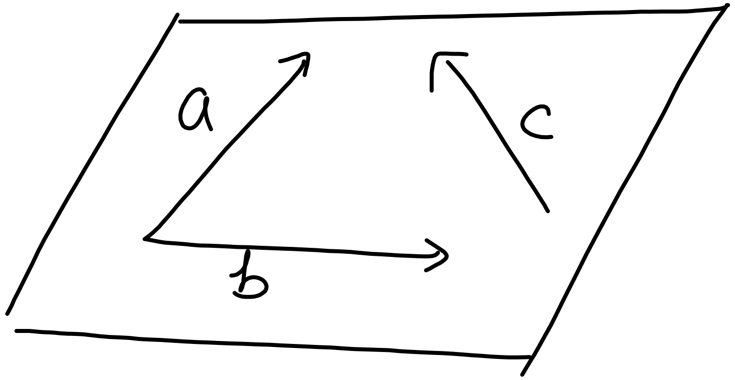
$\Rightarrow$ Any 2 vectors are always Coplanar

$\Rightarrow$ 3 vectors may or may not be Coplanar

How To check?

If 3 vectors are Coplanar

Volume of Parallelopied made by these 3 vectors $= 0$



$$[a\ b\ c] = 0$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$$

Q3) Find λ if the vectors $\vec{a} = \hat{\imath} + 3\hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{\jmath} - \hat{k}$ and $\vec{c} = \lambda\hat{\imath} + 7\hat{\jmath} + 3\hat{k}$ are coplanar

a) 1/2

b) 0

c) -1/2

d) -1

$\vec{a}, \vec{b}, \vec{c}$ Coplanar

$$[a\ b\ c] = 0$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 3 & 1 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= \hat{i}(-3+1) - \hat{j}(-1-2) + \hat{k}(-1-6)$$

$$(-2\hat{i} + 3\hat{j} - 7\hat{k}) \cdot (\lambda\hat{i} + 7\hat{j} + 3\hat{k}) = 0$$

$$-2\lambda + 21 - 21 = 0 \Rightarrow -2\lambda = 0 \quad \lambda = 0$$

Thank You

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