ELECTROSTATIC POTENTIAL ENERGY

Potential energy of a system of particles is defined only in conservative fields. As electric field is also conservative, we define potential energy in it. Potential energy of a system of particles we define as the work done in assembling the system in a given configuration against the interaction forces of particles. Electrostatic potential energy is defined in two ways.

(i) Interaction energy of charged particles of a system
(ii) Self energy of a charged object

- Electrostatic Interaction Energy

Electrostatic interaction energy of a system of charged particles is defined as the external work required to assemble the particles from infinity to the given configuration. When some charged particles are at infinite separation, their potential energy is taken zero as no interaction is there between them. When these charges are brought close to a given configuration, external work is required if the force between these particles is repulsive and energy is supplied to the system, hence final potential energy of system will be positive. If the force between the particle is attractive, work will be done by the system and final potential energy of system will be negative.
**Interaction Energy of a system of two charged particles**

Figure shows two + ve charges $q_1$ and $q_2$ separated by a distance $r$. The electrostatic interaction energy of this system can be given as work done in bringing $q_2$ from infinity to the given separation from $q_1$.

\[
W = \int \frac{F \cdot dx}{x} = -\int \frac{kq_1q_2}{x^2} dx \quad \{ - \text{ve sign shows that } x \text{ is decreasing} \}
\]

\[
W = \frac{kq_1q_2}{r} = U \quad \{ \text{interaction energy} \}
\]

If the two charges here are of opposite sign, the potential energy will be negative as $U = -\frac{kq_1q_2}{r}$

**Interaction Energy for a system of charged particles**

When more than two charged particles are there in a system, the interaction energy can be given by sum of interaction energies of all the pairs of particles. For example if a system of three particles having charges $q_1$, $q_2$ and $q_3$ is given as shown in figure.

The total interaction energy of this system can be given as $U = \frac{kq_1q_2}{r_3} + \frac{kq_1q_3}{r_2} + \frac{kq_2q_3}{r_1}$

**ELECTRIC POTENTIAL**

Electric potential is a scalar property of every point in the region of electric field. At a point in electric field potential is defined as the interaction energy of a unit positive charge. If at a point in electric field a charge $q_0$ has potential energy $U$, then electric potential at that point can be given as $V = \frac{U}{q_0}$ joule/coulomb

Potential energy of a charge in electric field is defined as work done in bringing the charge from infinity to the given point in electric field. Similarly we can define electric potential as "work done in bringing a unit positive charge from infinity to the given point against the electric forces.

**Example**

A charge $2\mu C$ is taken from infinity to a point in an electric field, without changing its velocity, if work done against electrostatic forces is $-40\mu J$ then potential at that point is ?

**Solution**

\[
V = \frac{W_{\text{ext}}}{q} = \frac{-40\mu J}{2\mu C} = -20V
\]

Note: Always remember to put sign of $W$ and $q$. 

---

**Note:** The diagram in the original text is not provided in this transcription. It would be helpful to have the diagram to fully understand the interaction between the charges. The description in the text refers to a diagram showing three charged particles forming a triangle, with distances $r_1$, $r_2$, and $r_3$ between them. The diagram is intended to illustrate the various forces and interactions between the charges, which are discussed in the text.
Electric Potential due to a point charge in its surrounding:

![Point Charge Diagram](image)

The potential at a point P at a distance r from the charge q is \( V_p = \frac{U}{q_0} \). Where U is the potential energy of charge q0 at point p, \( U = \frac{kq_0}{r} \). Thus potential at point P is \( V_p = \frac{kq}{r} \).

Electric Potential due to a charge Rod:

Figure shows a rod of length L, uniformly charged with a charge Q. Due to this we'll find electric potential at a point P at a distance r from one end of the rod as shown in figure.

![Charge Rod Diagram](image)

For this we consider an element of width \( dx \) at a distance x from the point P. Charge on this element is \( dQ = \frac{Q}{L} dx \).

The potential \( dV \) due to this element at point P can be given by using the result of a point charge as \( dV = \frac{kq}{x} = \frac{kQ}{Lx} dx \).

Net electric potential at point P: \( V = \int dV = \int_{r}^{L} \frac{kQ}{Lx} dx = \frac{kQ}{L} \ln \left( \frac{r+L}{r} \right) \).

Electric potential due to a charged ring

Case - I: At its centre

To find potential at the centre C of the ring, we first find potential \( dV \) at centre due to an elemental charge \( dq \) on ring which is given as \( dV = \frac{kq}{R} \). Total potential at C is \( V = \int dV = \int \frac{kq}{R} = \frac{kQ}{R} \).

![Charge Ring Diagram](image)

As all dq's of the ring are situated at same distance R from the ring centre C, simply the potential due to all dq's is added as being a scalar quantity, we can directly say that the total electric potential at ring centre is \( \frac{kQ}{R} \). Here we can also state that even if charge Q is non-uniformly distributed on ring, the electric potential C will remain same.
Case II: At a point on axis of ring

We find the electric potential at a point $P$ on the axis of ring as shown, we can directly state the result as here also all points of ring are at same distance $\sqrt{x^2 + R^2}$ from the point $P$, thus the potential at $P$ can be given as

$$V_p = \frac{kQ}{\sqrt{R^2 + x^2}}$$

Electric potential due to a uniformly charged disc:

Figure shows a uniformly disc of radius $R$ with surface charge density $\rho$ coul/m$^2$. To find electric potential at point $P$ we consider an elemental ring of radius $y$ and width $dy$, charge on this elemental ring is $dq = \sigma 2\pi y \, dy$.

Due to this ring, the electric potential at point $P$ can be given as

$$dV = \frac{k dq}{\sqrt{x^2 + y^2}} = \frac{k \sigma 2\pi y \, dy}{\sqrt{x^2 + y^2}}$$

Net electric potential at Point $P$ due to whole disc can be given as

$$V = \int dV = \int_0^\infty \frac{\sigma}{2 \epsilon_0} \frac{y \, dy}{\sqrt{x^2 + y^2}} = \frac{\sigma}{2 \epsilon_0} \left[ \sqrt{x^2 + y^2} \right]_0^\infty = \frac{\sigma}{2 \epsilon_0} \left[ \sqrt{x^2 + R^2} - x \right]$$

ELECTRIC POTENTIAL DUE TO HOLLOW OR CONDUCTING SPHERE

- At outside sphere

According to definition of electric potential, electric potential at point $P$

$$V = -\int \overrightarrow{E} \cdot d\overrightarrow{r} = -\int \frac{q}{4\pi \epsilon_0 r^2} \, dr \left[ \cdot E_{out} = \frac{q}{4\pi \epsilon_0 r^2} \right] \cdot V = -\frac{q}{4\pi \epsilon_0} \int \frac{1}{r} \, dr = \frac{q}{4\pi \epsilon_0} \left[ \frac{1}{r} \right] = \frac{q}{4\pi \epsilon_0 r}$$
At surface

\[ V = -\int_{-\infty}^{R} \textbf{E} \cdot d\mathbf{r} = -\int_{-\infty}^{R} \frac{q}{4\pi \varepsilon_0 r^2} dr \quad \Rightarrow \quad E_{\text{out}} = \frac{q}{4\pi \varepsilon_0 R^2} \]

\[ V = -\frac{q}{4\pi \varepsilon_0} \int_{-\infty}^{R} \left( \frac{1}{r} \right) dr = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{R} - \frac{1}{r} \right] \Rightarrow \quad V = \frac{q}{4\pi \varepsilon_0 R} \]

Inside the surface:

\[ \therefore \text{Inside the surface} \quad \frac{dV}{dr} = 0 \Rightarrow V = \text{constant} \quad \Rightarrow \quad V = \frac{q}{4\pi \varepsilon_0 R} \]

**ELECTRIC POTENTIAL DUE TO SOLID NON CONDUCTING SPHERE**

- At outside sphere: Same as conducting sphere.
- At surface: Same as conducting sphere.

Inside the sphere

\[ V = -\int_{-\infty}^{R} \textbf{E} \cdot d\mathbf{r} \quad \Rightarrow \quad V = -\left[ \int_{-\infty}^{R} \frac{1}{r} dr + \int_{R}^{r} \frac{kq}{r^2} dr \right] \]

\[ V = -\left[ \int_{-\infty}^{R} \frac{kq}{r^2} dr + \int_{R}^{r} \frac{kq r}{R^3} dr \right] \Rightarrow \quad V = -\left[ kq \left( \frac{1}{r} \right) + \frac{kq}{R^3} \left( \frac{r^2}{2} \right) \right] \]

\[ V = -kq \left[ 1 + \frac{r^2}{2R^2} - \frac{R^2}{2R^3} \right] \Rightarrow \quad V = \frac{kq}{2R^2} \left[ 3R^2 - r^2 \right] \]

**Potential Difference Between Two points in electric field**

Potential difference between two points in electric field can be defined as work done in displacing a unit charge from one point to another against the electric forces.

\[ \text{If a unit charge is displaced from a point A to B as shown work required can be given as } V_B - V_A = -\int_A^B \textbf{E} \cdot d\mathbf{r} \]

If a charge \( q \) is shifted from point A to B, work done against electric forces can be given as \( W = q \ (V_B - V_A) \)

If in a situation work done by electric forces is asked, we use \( W = q \ (V_A - V_B) \)

If \( V_B < V_A \), then charges must have tendency to move toward B (low potential point) it implies that electric forces carry the charge from high potential to low potential points. Hence we can say that in the direction of electric field always electric potential decreases.

**Example**

1\( \mu \)C charge is shifted from A to B and it is found that work done by external force is 80\( \mu \)J against electrostatic forces. Find \( V_A - V_B \)

**Solution**

\[ W_{\text{ext}} = q(V_B - V_A) \]

80\( \mu \)J = 1\( \mu \)C \((V_B - V_A) \Rightarrow V_A - V_B = -80 \text{ V} \)

**Equipotential surfaces**

For a given charge distribution, locus of all points having same potential is called 'equipotential surface'.

- Equipotential surfaces can never cross each other (otherwise potential at a point will have two values which is absurd)
• Equipotential surfaces are always perpendicular to the direction of the electric field.
• If a charge is moved from one point to the other over an equipotential surface then work done
  \[ W_{AB} = -U_{AB} = q \left( V_B - V_A \right) = 0 \quad \because \quad V_B = V_A \]
• Shapes of equipotential surfaces

![Diagram of equipotential surfaces](image)

• The intensity of electric field along an equipotential surface is always zero.

**Electric Potential Gradient**

The maximum rate of change of potential at right angles to an equipotential surface in an electric field is defined as the gradient of potential. 
\[ \vec{E} = -\nabla V = -\text{grad} \ V \]

**Note:** Potential is a scalar quantity but the gradient of potential is a vector quantity.

In Cartesian co-ordinates, \[ \nabla V = \left[ \frac{\partial V}{\partial x} \hat{i} + \frac{\partial V}{\partial y} \hat{j} + \frac{\partial V}{\partial z} \hat{k} \right] \]

**Example**

If \( V = -5x + 3y + \sqrt{15}z \), find the magnitude of the electric field at point \((x, y, z)\).

**Solution**

\[ \vec{E} = -\left[ \frac{\partial V}{\partial x} \hat{i} + \frac{\partial V}{\partial y} \hat{j} + \frac{\partial V}{\partial z} \hat{k} \right] = (-5 \hat{i} + 3 \hat{j} + \sqrt{15} \hat{k}) \Rightarrow |\vec{E}| = \sqrt{25 + 9 + 15} = \sqrt{49} = 7 \text{ unit} \]

**Example**

The four charges \( q \) each are placed at the corners of a square of side \( a \). Find the potential energy of one of the charges.

![Diagram of charges](image)

The electric potential of a point \( A \) due to charges \( B, C \) and \( D \) is

\[ V = \frac{1}{4\pi\varepsilon_0} \frac{q}{a} + \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{2}a} + \frac{1}{4\pi\varepsilon_0} \frac{q}{a} = \frac{1}{4\pi\varepsilon_0} \left( \frac{1}{2} + \frac{1}{\sqrt{2}} \right) \frac{q}{a} \]

\[ \therefore \quad \text{Potential energy of the charge at A is } PE = qV = \frac{1}{4\pi\varepsilon_0} \left( \frac{1}{2} + \frac{1}{\sqrt{2}} \right) \frac{q^2}{a}. \]
Example

A proton moves with a speed of \(7.45 \times 10^6\) m/s directly towards a free proton originally at rest. Find the distance of closest approach for the two protons.

Given: \(\frac{1}{4\pi \varepsilon_0} = 9 \times 10^9\) N \(\cdot\) m\(^2\) C\(^{-2}\); \(m_p = 1.67 \times 10^{-27}\) kg and \(e = 1.6 \times 10^{-19}\) C

Solution

As here the particle at rest is free to move, when one particle approaches the other, due to electrostatic repulsion other will also start moving and so the velocity of first particle will decrease while of other will increase and at closest approach both will move with same velocity. So if \(v\) is the common velocity of each particle at closest approach, by 'conservation of momentum'.

\[
mu = mv + mv \Rightarrow v = \frac{1}{2} u
\]

And by conservation of energy

\[
\frac{1}{2} mu^2 = \frac{1}{2} mv^2 + \frac{1}{2} + \frac{1}{4\pi \varepsilon_0} \frac{e^2}{r}
\]

So, \(r = \frac{4e^2}{4\pi \varepsilon_0 mu^2}\) [as \(v = \frac{u}{2}\)]

And hence substituting the given data,

\[
r = 9 \times 10^9 \frac{4 \times (1.6 \times 10^{-19})^2}{1.67 \times 10^{-27} \times (7.45 \times 10^3)^2} = 10^{-12}\) m

ELECTRIC DIPOLE

A system of two equal and opposite charges separated by a certain distance is called electric dipole, shown in figure. Every dipole has a characteristic property called dipole moment. It is defined as the product of magnitude of either charge and the separation between the charges, given as

\[
\vec{p} = q \vec{d}
\]

In some molecules, the centres of positive and negative charges do not coincide. This results in the formation of electric dipole. Atom is non-polar because in it the centres of positive and negative charges coincide. Polarity can be induced in an atom by the application of electric field. Hence it can be called as induced dipole.

- **Dipole Moment**: Dipole moment \(\vec{p} = q \vec{d}\)

  - (i) Vector quantity, directed from negative to positive charge
  - (ii) **Dimension**: [LTA], **Units**: coulomb metre (or C-m)
  - (iii) Practical unit is "debye" = Two equal and opposite point charges each having charge \(10^{-10}\) frankline (= e) and separation of 1Å then the value of dipole moment (\(\vec{p}\)) is 1 debye.

\[
1 \text{ Debye} = 10^{-10} 10^{-10} \text{ Fr m} = 10^{-20} \frac{C \times m}{3 \times 10^5} = 3.3 \times 10^{-30} \text{ C m}
\]
Example

A system has two charges \( q_A = 2.5 \times 10^{-7} \) C and \( q_B = -2.5 \times 10^{-7} \) C located at points A: \((0, 0, -0.15 \text{ m})\) and B: \((0, 0, +0.15 \text{ m})\) respectively. What is the total charge and electric dipole moment of the system?

Solution

Total charge = \( 2.5 \times 10^{-7} - 2.5 \times 10^{-7} = 0 \)

Electric dipole moment,

\[ p = \text{Magnitude of either charge separation between charges} \]

\[ = 2.5 \times 10^{-7} \times |0.15 + 0.15| \text{ C m} = 7.5 \times 10^{-8} \text{ C m.} \] The direction of dipole moment is from B to A.

Dipole Placed in uniform Electric Field

Figure shows a dipole of dipole moment \( \vec{p} \) placed at an angle \( \theta \) to the direction of electric field. Here the charges of dipole experience forces \( qE \) in opposite direction as shown. \( \vec{F}_{\text{net}} = [qE + (-q)\vec{E}] = \vec{0} \)

Thus we can state that when a dipole is placed in a uniform electric field, net force on the dipole is zero. But as equal and opposite forces act with a separation in their line of action, they produce a couple which tend to align the dipole along the direction of electric field. The torque due to this couple can be given as

\[ \tau = \text{Force separation between lines of actions of forces} = qE \times d \sin \theta = pE \sin \theta \]

\[ \tau = \vec{r} \times \vec{F} = \vec{d} \times qE = q \vec{d} \times \vec{E} = \vec{p} \times \vec{E} \]

Work done in Rotation of a Dipole in Electric field

When a dipole is placed in an electric field at an angle \( \theta \), the torque on it due to electric field is \( \tau = pE \sin \theta \)

Work done in rotating an electric dipole from \( \theta_1 \) to \( \theta_2 \) [uniform field]

\[ dW = \tau d\theta \] so \[ W = \int dW = \int \tau d\theta \] and \[ W_{\theta_1 \rightarrow \theta_2} = W = \int_{\theta_1}^{\theta_2} pE \sin \theta \, d\theta = pE (\cos \theta_1 - \cos \theta_2) \]

E.g. \( W_{0 \rightarrow 180} = pE (1 - (-1)) = 2pE \) \( W_{0 \rightarrow 90} = pE (1-0) = pE \)

If a dipole is rotated from field direction (\( \theta = 0 \)) to \( 0 \) then \( W = pE (1 - \cos \theta) \)

\[ \theta = 0 \quad \theta = 90 \quad \theta = 180 \]
\[ \tau = \text{minimum} = 0 \quad \tau = \text{maximum} = pE \quad \tau = \text{minimum} = 0 \]
\[ W = \text{minimum} = 0 \quad W = pE \quad W = \text{maximum} = 2pE \]
Electrostatic potential energy:

Electrostatic potential energy of a dipole placed in a uniform field is defined as work done in rotating a dipole from a direction perpendicular to the field to the given direction i.e., \( W_{\text{rot}} = \int_0^\theta pE \sin \theta \, d\theta = -pE \cos \theta = -p\vec{E} \).

\( \vec{E} \) is a conservative field so whatever work is done in rotating a dipole from \( \theta_1 \) to \( \theta_2 \) is just equal to change in electrostatic potential energy \( W_{\theta_1 \rightarrow \theta_2} = U_{\theta_2} - U_{\theta_1} = pE \cos \theta_1 - \cos \theta_2 \).

Work done in rotating an electric dipole in an electric field:

Suppose at any instant, the dipole makes an angle \( \theta \) with the electric field. The torque acting on dipole, \( \tau = q2l \sin \theta \vec{E} = pE \sin \theta \). The work done in rotating dipole from \( \theta_1 \) to \( \theta_2 \)

\[
W = \int_{\theta_1}^{\theta_2} \tau d\theta = \int_{\theta_1}^{\theta_2} pE \sin \theta \, d\theta
\]

\[
W = pE (\cos \theta_1 - \cos \theta_2) = U_2 - U_1 \quad (\therefore U = -pE \cos \theta)
\]

Force on an electric dipole in Non-uniform electric field:

If in a non-uniform electric field dipole is placed at a point where electric field is \( \vec{E} \), the interaction energy of dipole at this point \( U = -p\vec{E} \). Now the force on dipole due to electric field \( F = -\frac{\Delta U}{\Delta r} \).

If dipole is placed in the direction of electric field then \( F = -p \frac{dE}{dr} \).

Example:

Calculate force on a dipole in the surrounding of a long charged wire as shown in the figure.

Solution:

In the situation shown in figure, the electric field strength due to the wire, at the position of dipole as \( E = \frac{2k\lambda}{r} \).

Thus force on dipole is \( F = -p \frac{dE}{dr} = -p \left[ -\frac{2k\lambda}{r^2} \right] = \frac{2kp\lambda}{r^2} \).

Here -ve charge of dipole is close to wire hence net force an dipole due to wire will be attractive.
ELECTRIC POTENTIAL DUE TO DIPOLE

- At axial point

Electric potential due to $+q$ charge $V_1 = \frac{kq}{r - l}$

Electric potential due to $-q$ charge $V_2 = \frac{-kq}{r + l}$

Net electric potential $V = V_1 + V_2 = \frac{kq}{r - l} + \frac{-kq}{r + l} = \frac{kq \times 2l}{r^2 - l^2} = \frac{kp}{r^2}$

If $r >> \ell \Rightarrow V = \frac{kp}{r^2}$

- At equatorial point

Electric potential of P due to $+q$ charge $V_1 = \frac{kq}{x}$

Electric potential of P due to $-q$ charge $V_2 = \frac{-kq}{x}$

Net potential $V = V_1 + V_2 = \frac{kq}{x} - \frac{kq}{x} = 0 \therefore V = 0$

- At general point

$V = \frac{p \cos \theta}{4 \pi \epsilon_0 r^2} = \frac{\vec{p} \cdot \vec{r}}{4 \pi \epsilon_0 r^3}$, $\vec{p} = q \vec{d}$ electric dipole moment

Electric field due to an electric dipole

Figure shows an electric dipole placed on x-axis at origin. Here we wish to find the electric field and potential at a point O having coordinates $(r, \theta)$. Due to the positive charge of dipole electric field at O is in radially outward direction and due to the negative charge it is radially inward as shown in figure.

$E_r = \frac{\partial V}{\partial r} = \frac{2kp \cos \theta}{r^3}$ and $E_\theta = -\frac{1}{r} \frac{\partial V}{\partial \theta} = \frac{kp \sin \theta}{r^3}$

Thus net electric field at point O, \( E_{net} = \sqrt{E_r^2 + E_\theta^2} = \frac{kp}{r^3} \sqrt{1 + 3 \cos^2 \theta} \)

If the direction of \( E_{net} \) is at an angle $\alpha$ from radial direction, then $\alpha = \tan^{-1} \left( \frac{E_\theta}{E_r} \left( \frac{1}{2} \tan \theta \right) \right)$

Thus the inclination of net electric field at point O is $(\theta + \alpha)$
At a point on the axis of a dipole:

Electric field due to +q charge $E_1 = \frac{kq}{(r-\ell)^2}$

Electric field due to −q charge $E_2 = \frac{kq}{(r+\ell)^2}$

Net electric field $E = E_1 - E_2 = \frac{kq}{(r-\ell)^2} - \frac{kq}{(r+\ell)^2} = \frac{kq \times 4\ell}{(r^2 - \ell^2)^2}$ \[\therefore p = q \quad 2\ell = \text{Dipole moment}\]

$E = \frac{2kr}{(r^2 - \ell^2)^2}$ \[\text{If } r \gg \ell \quad \text{then } E = \frac{2kp}{r^3}\]

At a point on equatorial line of dipole:

Electric field due to +q charge $E_1 = \frac{kq}{x^2}$; Electric field due to −q charge $E_2 = \frac{kq}{x^2}$

Vertical component of $E_1$ and $E_2$ will cancel each other and horizontal components will be added. So net electric field at P

$E = E_1 \cos \theta + E_2 \cos \theta$ \[\therefore E_1 = E_2\]

$E = 2E_1 \cos \theta = \frac{2kq}{x^2} \cos \theta$ \[\therefore \cos \theta = \frac{\ell}{x} \text{ and } x = \sqrt{r^2 + \ell^2}\]

$E = \frac{2k\ell}{x^3} = \frac{2k\ell}{(r^2 + \ell^2)^{3/2}} = \frac{kp}{r^3}$ \[\text{If } r \gg \ell \quad \text{then } E = \frac{kp}{r^3} \text{ or } \vec{E} = -\frac{kp}{r^3}\]

GOLDEN KEY POINTS

For a dipole, potential is zero at equatorial position, while at any finite point $E \neq 0$

In a uniform $\vec{E}$, dipole may feel a torque but not a force.

If a dipole placed in a field $\vec{E}$ (Non-Uniform) generated by a point charge, then torque on dipole may be zero, but $F \neq 0$

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Example

A short electric dipole is situated at the origin of coordinate axis with its axis along x-axis and equator along y-axis. It is found that the magnitudes of the electric intensity and electric potential due to the dipole are equal at a point distant $r = \sqrt{5}$ m from origin. Find the position vector of the point in first quadrant.

Solution

\[\therefore |E_p| = |V_p| \therefore \frac{kp}{r^3} \sqrt{1 + 3 \cos^2 \theta} = \frac{kp \cos \theta}{r^2} \Rightarrow 1 + 3 \cos^2 \theta = 5 \cos^2 \theta \Rightarrow \cos \theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ\]

Position vector $\vec{r}$ of point P is

$\vec{r} = \frac{\sqrt{5}}{2} (\hat{i} + \hat{j})$
Example

Prove that the frequency of oscillation of an electric dipole of moment \( p \) and rotational inertia \( I \) for small amplitudes about its equilibrium position in a uniform electric field strength \( E \) is

\[
\omega = \frac{1}{2\pi} \sqrt{\frac{pE}{I}}
\]

Solution

Let an electric dipole (charge \( q \) and \(-q\) at a distance \( 2a \) apart) placed in a uniform external electric field of strength \( E \).

Restoring torque on dipole

\[
\tau = -pE\sin\theta = -pE\theta \quad \text{(as } \theta \text{ is small)}
\]

Here -ve sign shows the restoring tendency of torque. \( \therefore \tau = I\alpha \therefore \) angular acceleration \( \alpha = \frac{\tau}{I} = \frac{pE}{I} \theta \)

For SHM \( \alpha = -\omega^2\theta \) comparing we get \( \omega = \sqrt{\frac{pE}{I}} \)

Thus frequency of oscillations of dipole \( n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{pE}{I}} \)

ELECTROSTATIC PRESSURE

Force due to electrostatic pressure is directed normally outwards to the surface.

Force on small element \( ds \) of charged conductor

\[
dF = (\text{Charge on } ds) \times \text{Electric field} = (\sigma \, ds) \frac{\sigma}{2\varepsilon_0} = \frac{\sigma^2}{2\varepsilon_0} \, ds
\]

Inside \( E_1 - E_2 = 0 \Rightarrow E_1 = E_2 \)

Just outside \( E = E_1 + E_2 = 2E_2 \Rightarrow E_2 = \frac{\sigma}{2\varepsilon_0} \)

(\( E_1 \) is field due to point charge on the surface and \( E_2 \) is field due to rest of the sphere).

The electric force acting per unit area of charged surface is defined as electrostatic pressure.

\[
P_{\text{electrostatic}} = \frac{dF}{dS} = \frac{\sigma^2}{2\varepsilon_0}
\]
Equilibrium of liquid charged surfaces (Soap bubble)

Pressures (forces) act on a charged soap bubble, due to

(i) Surface tension $P_t$ (inward)
(ii) Air outside the bubble $P_o$ (inward)
(iii) Electrostatic pressure $P_e$ (outward)
(iv) Air inside the bubble $P_i$ (outward)

in state of equilibrium inward pressure = outward pressure $P_t + P_o = P_i + P_e$

Excess pressure of air inside the bubble $(P_a) = P_i - P_o = P_t - P_e$

but $P_t = \frac{4T}{r}$ and $P_e = \frac{\sigma^2}{2\epsilon_0} \Rightarrow P_a = \frac{4T}{r} - \frac{\sigma^2}{2\epsilon_0}$ if $P_i = P_o$, then $\frac{4T}{r} = \frac{\sigma^2}{2\epsilon_0}$

Example

Brass has a tensile strength $3.5 \times 10^6$ N/m$^2$. What charge density on this material will be enough to break it by electrostatic force of repulsion? How many excess electrons per square Å will there then be? What is the value of intensity just outside the surface?

Solution

We know that electrostatic force on a charged conductor is given by $\frac{dF}{ds} = \frac{\sigma^2}{2\epsilon_0}$

So the conductor will break by this force if $\frac{\sigma^2}{2\epsilon_0} >$ Breaking strength i.e., $\sigma^2 > 2 \times 9 \times 10^{-12} \times 3.5 \times 10^6$

i.e. $\sigma_{\text{min}} = (3\sqrt{7}) \times 10^{-2} = 7.94 \times 10^{-2}$ (C/m$^2$)

Now as the charge on an electron is $1.6 \times 10^{-19}$ C, the excess electrons per m$^2$

Further as in case of a conductor near its surface $E = \frac{\sigma}{\epsilon_0} = \frac{7.94 \times 10^{-2}}{9 \times 10^{-12}} = 8.8 \times 10^9$ V/m

CONDUCTOR AND IT'S PROPERTIES [FOR ELECTROSTATIC CONDITION]

(i) Conductors are materials which contain large number of free electrons which can move freely inside the conductor.

(ii) In electrostatics, conductors are always equipotential surfaces.

(iii) Charge always resides on outer surface of conductor.

(iv) If there is a cavity inside the conductor having no charge then charge will always reside only on outer surface of conductor.

(v) Electric field is always perpendicular to conducting surface.

(vi) Electric lines of force never enter into conductors.

(vii) Electric field intensity near the conducting surface is given by formula $E = \frac{\sigma}{\epsilon_0} \hat{n}$

\[ \vec{E}_A = \frac{\sigma_A}{\epsilon_0} \hat{n} ; \vec{E}_B = \frac{\sigma_B}{\epsilon_0} \hat{n} \text{ and } \vec{E}_C = \frac{\sigma_C}{\epsilon_0} \hat{n} \]

(viii) When a conductor is grounded its potential becomes zero.
(ix) When an isolated conductor is grounded then its charge becomes zero.

(x) When two conductors are connected there will be charge flow till their potential becomes equal.

(xi) Electric pressure at the surface of a conductor is given by formula \[ P = \frac{\sigma^2}{2\varepsilon_0} \] where \( \sigma \) is the local surface charge density.

Example

Prove that if an isolated (isolated means no charges are near the sheet) large conducting sheet is given a charge then the charge distributes equally on its two surfaces.

Solution

Let there be \( x \) charge on left side of sheet and \( Q-x \) charge on right side of sheet.

Since point \( P \) lies inside the conductor so \( E_p = 0 \)

\[
\frac{x}{2A\varepsilon_0} - \frac{Q-x}{2A\varepsilon_0} = 0 \Rightarrow \frac{2x}{2A\varepsilon_0} = \frac{Q}{2A\varepsilon_0} \Rightarrow x = \frac{Q}{2}
\]

\[ Q-x = \frac{Q}{2} \]

So charge is equally distributed on both sides.

Example

If an isolated infinite sheet contains charge \( Q_1 \) on its one surface and charge \( Q_2 \) on its other surface then prove that electric field intensity at a point in front of sheet will be \[ \frac{Q}{2A\varepsilon_0} \], where \( Q = Q_1 + Q_2 \)

Solution

Electric field at point \( P \):

\[ \mathbf{E} = \mathbf{E}_{Q_1} + \mathbf{E}_{Q_2} = \frac{Q_1}{2A\varepsilon_0} \hat{n} + \frac{Q_2}{2A\varepsilon_0} \hat{n} = \frac{Q_1 + Q_2}{2A\varepsilon_0} \hat{n} = \frac{Q}{2A\varepsilon_0} \hat{n} \]

\[ Q_1 \quad Q_2 \]

\[ \hat{n} \]

\[ P \]

\[ \frac{Q_1}{2A\varepsilon_0} + \frac{Q_2}{2A\varepsilon_0} \]

[This shows that the resultant field due to a sheet depends only on the total charge of the sheet and not on the distribution of charge on individual surfaces].

Example

Three large conducting sheets placed parallel to each other at finite distance contains charges \( Q \), \(-2Q\) and \(3Q\) respectively. Find electric field at points A, B, C, and D.

\[
A \quad B \quad C \quad D
\]

\[
Q \quad -2Q \quad 3Q
\]

Sol. \( E_A = E_Q + E_{-2Q} + E_{3Q} \). (i) Here \( E_Q \) means electric field due to 'Q'.

\[
E_A = \frac{(Q-2Q+3Q)}{2A\varepsilon_0} = \frac{2Q}{2A\varepsilon_0} = \frac{Q}{A\varepsilon_0}, \text{ towards left}
\]
(ii) \[ E_B = \frac{Q - (-2Q + 3Q)}{2A\varepsilon_0}, \text{ towards right } = 0 \]

(iii) \[ E_C = \frac{(Q - 2Q) - (3Q)}{2A\varepsilon_0} = \frac{-4Q}{2A\varepsilon_0} = \frac{-2Q}{A\varepsilon_0}, \text{ towards right } \Rightarrow \frac{2Q}{A\varepsilon_0}, \text{ towards left} \]

(iv) \[ E_D = \frac{(Q - 2Q + 3Q)}{2A\varepsilon_0} = \frac{2Q}{2A\varepsilon_0} = \frac{Q}{A\varepsilon_0}, \text{ towards right} \]

Example

Two conducting plates A and B are placed parallel to each other. A is given a charge \( Q_1 \) and B a charge \( Q_2 \). Prove that the charges on the inner facing surfaces are of equal magnitude and opposite sign. Also find the charges on inner & outer surfaces.

Solution

Consider a Gaussian surface as shown in figure. Two faces of this closed surface lie completely inside the conductor where the electric field is zero. The flux through these faces is, therefore, zero. The other parts of the closed surface which are outside the conductor are parallel to the electric field and hence the flux on these parts is also zero. The total flux of the electric field through the closed surface is, therefore zero. From Gauss’s law, the total charge inside this closed surface should be zero. The charge on the inner surface of A should be equal and opposite to that on the inner surface of B.

The distribution should be like the one shown in figure. To find the value of \( q \), consider the field at a point P inside the plate A. Suppose, the surface area of the plate (one side) is A. Using the equation \( E = \frac{\sigma}{2A\varepsilon_0} \), the electric field at P

due to the charge \( Q_1 - q = \frac{Q_1 - q}{2A\varepsilon_0} \) (downward); due to the charge \( +q = \frac{q}{2A\varepsilon_0} \) (upward),

due to the charge \( -q = \frac{q}{2A\varepsilon_0} \) (downward), and due to the charge \( Q_2 + q = \frac{Q_2 + q}{2A\varepsilon_0} \) (upward).

The net electric field at P due to all the four charged surfaces is (in the downward direction)

\[ E_p = \frac{Q_1 - q}{2A\varepsilon_0} - \frac{q}{2A\varepsilon_0} + \frac{q}{2A\varepsilon_0} - \frac{Q_2 + q}{2A\varepsilon_0} \]

As the point P is inside the conductor, this field should be zero.

Hence, \( Q_1 - q - q + q - Q_2 - q = 0 \Rightarrow q = \frac{Q_1 - Q_2}{2} \)

This result is a special case of the following result. When charged conducting plates are placed parallel to each other, the two outermost, surfaces get equal charges and the facing surfaces get equal and opposite charges.
Example

Figure shows three large metallic plates with charges $-Q$, $3Q$ and $Q$ respectively. Determine the final charges on all the surfaces.

Solution

We assume that charge on surface 2 is $x$. Following conservation of charge, we see that surfaces 1 has charge $(-Q - x)$. The electric field inside the metal plate is zero so fields at $P$ is zero.

Resultant field at $P$ : $E_r = 0 \Rightarrow \frac{-Q - x}{2\varepsilon_0} = \frac{x + 3Q + Q}{2\varepsilon_0} \Rightarrow -Q - x = x + 4Q \Rightarrow x = -\frac{5Q}{2}$

Note: We see that charges on the facing surfaces of the plates are of equal magnitude and opposite sign. This can be in general proved by gauss theorem also. Remember this it is important result.

Thus the final charge distribution on all the surfaces is:

Example

An isolated conducting sheet of area $A$ and carrying a charge $Q$ is placed in a uniform electric field $E$, such that electric field is perpendicular to sheet and covers all the sheet. Find out charges appearing on its two surfaces. Also
Solution

Let there is \( x \) charge on left side of plate and \( Q - x \) charge on right side of plate

\[
E_0 = 0 \Rightarrow \frac{x}{2A\varepsilon_0} + E = \frac{Q - x}{2A\varepsilon_0} \Rightarrow \frac{x}{A\varepsilon_0} = \frac{Q}{2A\varepsilon_0} - E \Rightarrow x = \frac{Q}{2} - EA\varepsilon_0
\]

So charge on one side is \( \frac{Q}{2} - EA\varepsilon_0 \) and other side \( \frac{Q}{2} + EA\varepsilon_0 \).

The resultant electric field on the left and right side of the plate.

On right side \( E = \frac{Q}{2A\varepsilon_0} + E \) towards right and on left side \( \frac{Q}{2A\varepsilon_0} - E \) towards left.

**SOME IMPORTANT RESULTS FOR A CLOSED CONDUCTOR.**

(i) If a charge \( q \) is kept in the cavity then \( -q \) will be induced on the inner surface and \( +q \) will be induced on the outer surface of the conductor (it can be proved using Gauss theorem)

(ii) If a charge \( q \) is kept inside the cavity of a conductor and conductor is given a charge \( Q \) then \( -q \) charge will be induced on inner surface and total charge on the outer surface will be \( q + Q \). (It can be proved using Gauss theorem)

(iii) Resultant field, due to \( q \) (which is inside the cavity) and induced charge on \( S_1 \), at any point outside \( S_1 \) (like \( B, C \)) is zero. Resultant field due to \( q + Q \) on \( S_2 \) and any other charge outside \( S_2 \), at any point inside of surface \( S_2 \) (like \( A, B \)) is zero
Resultant field in a charge free cavity in a closed conductor is zero. There can be charges outside the conductor and on the surface also. Then also this result is true. No charge will be induced on the inner most surface of the conductor.

![Diagram showing charge distribution in conductors](image)

Charge distribution for different types of cavities in conductors

(A) charge is at the common centre \( (S_1, S_2 \rightarrow \text{spherical}) \)

(B) charge is not at the common centre \( (S_1, S_2 \rightarrow \text{spherical}) \)

(C) charge is at the centre of \( S_2 \) \( (S_2 \rightarrow \text{spherical}) \)

(D) charge is not at the centre of \( S_2 \) \( (S_2 \rightarrow \text{spherical}) \)

(E) charge is at the centre of \( S_1 \) (Spherical)

(F) charge not at the centre of \( S_1 \) (Spherical)

Using the result that \( \vec{E}_{\text{res}} \) in the conducting material should be zero and using result (iii) We can show that

<table>
<thead>
<tr>
<th>Case</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1 )</td>
<td>Uniform</td>
<td>Nonuniform</td>
<td>Nonuniform</td>
<td>Nonuniform</td>
<td>Uniform</td>
<td>Nonuniform</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>Uniform</td>
<td>Uniform</td>
<td>Uniform</td>
<td>Uniform</td>
<td>Nonuniform</td>
<td>Nonuniform</td>
</tr>
</tbody>
</table>

Note: In all cases charge on inner surface \( S_1 = -q \) and on outer surface \( S_2 = q \). The distribution of charge on \( S_1 \) will not change even if some charges are kept outside the conductor (i.e. outside the surface \( S_2 \)). But the charge distribution on \( S_2 \) may change if some charges(s) is/are kept outside the conductor.
Example

An uncharged conductor of inner radius $R_1$ and outer radius $R_2$ contains a point charge $q$ at the centre as shown in figure.

(i) Find $\vec{E}$ and $V$ at points $A$, $B$ and $C$.

(ii) If a point charge $Q$ is kept outside the sphere at a distance $r$ ($r \gg R_2$) from the centre then find out the resultant force on charge $Q$ and charge $q$.

Solution

At point $A$: $V_A = \frac{Kq}{OA} + \frac{K(-q)}{R_1} + \frac{Kq}{R_2}$, $\vec{E}_A = \frac{Kq}{OA^3} \hat{OA}$

At point $B$: $V_B = \frac{Kq}{OB} + \frac{K(-q)}{OB} + \frac{Kq}{R_2}$, $E_B = 0$; At point $C$: $V_C = \frac{Kq}{OC}$, $\vec{E}_C = \frac{Kq}{OC^3} \hat{OC}$

(ii) Force on point charge $Q$: $\vec{F}_Q = \frac{KqQ}{r^2} \hat{r}$ ($r =$ distance of 'Q' from centre 'O')

Force on point charge $q$: $\vec{F}_q = 0$ (using result (iii) & charge on $S_1$ uniform)

Example

An uncharged conductor of inner radius $R_1$ and outer radius $R_2$ contains a point charge $q$ placed at point $P$ (not at the centre) as shown in figure. Find out the following:

(i) $V_C$  (ii) $V_A$  (iii) $V_B$  (iv) $E_A$  (v) $E_B$

(vi) force on charge $Q$ if it is placed at $B$.

Solution

(i) $V_C = \frac{Kq}{CP} + \frac{K(-q)}{R_1} + \frac{Kq}{R_2}$

(ii) $V_A = \frac{Kq}{R_2}$

(iii) $V_B = \frac{Kq}{CB}$

(iv) $E_A = 0$ (point is inside metallic conductor)

(v) $E_B = \frac{Kq}{CB^3} \hat{CB}$

(vi) $F_Q = \frac{KqQ}{CB^3} \hat{CB}$
(vi) **Sharing of charges:**

Two conducting hollow spherical shells of radii \( R_1 \) and \( R_2 \) having charges \( Q_1 \) and \( Q_2 \) respectively and separated by large distance, are joined by a conducting wire. Let final charges on spheres are \( q_1 \) and \( q_2 \) respectively.

![Diagram](image)

Potential on both spherical shell become equal after joining, therefore

\[
\frac{Kq_1}{R_1} = \frac{Kq_2}{R_2}; \quad q_1 = \frac{R_1}{R_2} \quad \text{and} \quad q_1 + q_2 = Q_1 + Q_2 \quad \text{......(ii)}
\]

From (i) and (ii) \( q_1 = \frac{(Q_1 + Q_2)R_1}{R_1 + R_2} \); \( q_2 = \frac{(Q_1 + Q_2)R_2}{R_1 + R_2} \)

ratio of charges \( \frac{q_1}{q_2} = \frac{R_1}{R_2} \); \( \frac{\sigma_1 4\pi R_1^2}{\sigma_2 4\pi R_2^2} = \frac{R_1}{R_2} \)

ratio of surface charge densities \( \frac{\sigma_1}{\sigma_2} = \frac{R_2}{R_1} \)

**Ratio of final charges**

\( \frac{q_1}{q_2} = \frac{R_1}{R_2} \)

**Ratio of final surface charge densities.**

\( \frac{\sigma_1}{\sigma_2} = \frac{R_2}{R_1} \)

**Example**

The two conducting spherical shells are joined by a conducting wire and cut after some time when charge stops flowing. Find out the charge on each sphere after that.

![Diagram](image)

**Solution**

After cutting the wire, the potential of both the shells is equal

Thus, potential of inner shell \( V_{in} = \frac{Kx}{R} + \frac{K(-2Q-x)}{2R} = \frac{k(x-2Q)}{2R} \)

and potential of outer shell \( V_{out} = \frac{Kx}{2R} + \frac{K(-2Q-x)}{2R} = -\frac{KQ}{R} \)

As \( V_{out} = V_{in} \Rightarrow \frac{-KQ}{R} = \frac{K(x-2Q)}{2R} \Rightarrow -2Q = x - 2Q \Rightarrow x = 0 \)

So charge on inner spherical shell = 0 and outer spherical shell = \(-2Q\).
Example

Find charge on each spherical shell after joining the inner most shell and outer most shell by a conducting wire. Also find charges on each surface.

Solution

Let the charge on the innermost sphere be \( x \).

Finally potential of shell 1 = Potential of shell 3

\[
\frac{Kx}{R} + \frac{K(-2Q)}{2R} + \frac{K(6Q-x)}{3R} = \frac{KQ}{3R} + \frac{k(-2q)}{3R} + \frac{k(5Q)}{3R}
\]

\[
x - 3Q + 6Q - x = 4Q ; 2x = Q ; \quad x = \frac{Q}{2}
\]

Charge on innermost shell = \( \frac{Q}{2} \), charge on outermost shell = \( \frac{5Q}{2} \)

middle shell = \( -2Q \)

Final charge distribution is as shown in figure.

Example

Two conducting hollow spherical shells of radii \( R \) and \( 2R \) carry charges \( Q \) and \( 3Q \) respectively. How much charge will flow into the earth if inner shell is grounded?

Solution

When inner shell is grounded to the Earth then the potential of inner shell will become zero because potential of the Earth is taken to be zero.

\[
\frac{Kx}{R} + \frac{K3Q}{2R} = 0
\]

\[
x = \frac{-3Q}{2}, \quad \text{the charge that has increased}
\]

\[
= \frac{-3Q}{2} - (-Q) = \frac{Q}{2} \quad \text{hence charge flows into the Earth} = \frac{Q}{2}
\]

Example

An isolated conducting sphere of charge \( Q \) and radius \( R \) is connected to a similar uncharged sphere (kept at a large distance) by using a high resistance wire. After a long time what is the amount of heat loss?

Solution

When two conducting spheres of equal radius are connected charge is equally distributed on them.

So we can say that heat loss of system

\[
\Delta H = U_i - U_f = \left( \frac{Q^2}{8\pi \varepsilon_0 R} - 0 \right) - \left( \frac{Q^2}{8\pi \varepsilon_0 R} + \frac{Q^2}{8\pi \varepsilon_0 R} + \frac{Q^2}{8\pi \varepsilon_0 R} \right) = \frac{Q^2}{16\pi \varepsilon_0 R}
\]
SOME WORKED OUT EXAMPLES

Example #1
For a spherically symmetrical charge distribution, electric field at a distance \( r \) from the centre of sphere is \( \vec{E} = kr^2 \hat{r} \), where \( k \) is a constant. What will be the volume charge density at a distance \( r \) from the centre of sphere? 
(A) \( \rho = 9k\varepsilon_0 r^6 \)  
(B) \( \rho = 5k\varepsilon_0 r^3 \)  
(C) \( \rho = 3k\varepsilon_0 r^4 \)  
(D) \( \rho = 9k\varepsilon_0 r^0 \)

Solution
By using Gauss law \( \oint \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0} \Rightarrow (E)(4\pi r^2) = \frac{\int \rho (4\pi r^2 dr)}{\varepsilon_0} \)
(Note: Check dimensionally that \( \rho \propto r^6 \))
\( (kr^2)(4\pi r^2) = \frac{\int \rho (4\pi r^2 dr)}{\varepsilon_0} \Rightarrow k\varepsilon_0 r^6 = \int \rho r^2 dr \)

Example #2
Two positrons (e⁺) and two protons (p) are kept on four corners of a square of side \( a \) as shown in figure. The mass of proton is much larger than the mass of positron. Let \( q \) denotes the charge on the proton as well as the positron then the kinetic energies of one of the positrons and one of the protons respectively after a very long time will be–

(A) \( \frac{q^2}{4\pi \varepsilon_0 a} \left( 1 + \frac{1}{2\sqrt{2}} \right), \frac{q^2}{4\pi \varepsilon_0 a} \left( 1 + \frac{1}{2\sqrt{2}} \right) \)  
(B) \( \frac{q^2}{2\pi \varepsilon_0 a}, \frac{q^2}{4\sqrt{2} \pi \varepsilon_0 a} \)

(C) \( \frac{q^2}{4\pi \varepsilon_0 a}, \frac{q^2}{4\pi \varepsilon_0 a} \)  
(D) \( \frac{q^2}{2\pi \varepsilon_0 a}, \frac{q^2}{4\sqrt{2} \pi \varepsilon_0 a} \)

Solution
As mass of proton \( \gg \gg \) mass of positron so initial acceleration of positron is much larger than proton. Therefore positron reach far away in very short time as compare to proton.

\[ 2K_{e^+} = \left( \frac{4kq^2}{a} + \frac{2kq^2}{a\sqrt{2}} \right) - \frac{kq^2}{a\sqrt{2}} \Rightarrow K_{e^+} = \frac{q^2}{2\pi \varepsilon_0 a} \left( 1 + \frac{1}{4\sqrt{2}} \right) \]  
and \[ 2K_p = \frac{kq^2}{a\sqrt{2}} - 0 \Rightarrow K_p = \frac{q^2}{8\sqrt{2} \pi \varepsilon_0 a} \]
Example #3

Four charges are placed at the circumference of a dial clock as shown in figure. If the clock has only hour hand, then the resultant force on a charge $q_c$ placed at the centre, points in the direction which shows the time as:

(A) 1:30  (B) 7:30  (C) 4:30  (D) 10:30

Solution

Ans. (B)

Example #4

A small electric dipole is placed at origin with its dipole moment directed along positive x-axis. The direction of electric field at point $(2, 2\sqrt{2}, 0)$ is

(A) along z-axis  (B) along y-axis  (C) along negative y-axis  (D) along negative z-axis

Solution

Ans. (B)

\[
\tan \theta = \frac{y}{x} = \frac{\sqrt{2}}{2}; \cot \theta = \frac{1}{\sqrt{2}} \quad \text{Also} \quad \tan \alpha = \frac{\tan \theta}{2} = \frac{1}{\sqrt{2}} = \cot \theta \Rightarrow \theta + \alpha = 90 \quad \text{i.e.,} \quad E \text{ is along positive y-axis.}
\]

Example #5

Uniform electric field of magnitude 100 V/m in space is directed along the line $y = 3 + x$. Find the potential difference between point A $(3, 1)$ & B $(1,3)$.

(A) 100 V  (B) 200V√2V  (C) 200 V  (D) zero

Solution

Ans. (D)

Slope of line AB $= \frac{3 - 1}{1 - 3} = -1$ which is perpendicular to direction of electric field.
Example #6
The diagram shows a uniformly charged hemisphere of radius R. It has volume charge density \( \rho \). If the electric field at a point 2R distance above its centre is \( E \) then what is the electric field at the point which is 2R below its centre?

\[ (A) \frac{\rho R}{6 \varepsilon_0} + E \quad (B) \frac{\rho R}{12 \varepsilon_0} - E \quad (C) \frac{-\rho R}{6 \varepsilon_0} + E \quad (D) \frac{\rho R}{24 \varepsilon_0} + E \]

**Solution**

\[ \text{Apply principle of superposition} \]

Electric field due to a uniformly charged sphere = \( \frac{\rho R}{12 \varepsilon_0} \); \( E_{\text{resultant}} = \frac{\rho R}{12 \varepsilon_0} - E \)

**Example #7**
A metallic rod of length \( l \) rotates at angular velocity \( \omega \) about an axis passing through one end and perpendicular to the rod. If mass of electron is \( m \) and its charge is \(-e\) then the magnitude of potential difference between its two ends is

\[ (A) \frac{m \omega^2 l^2}{2e} \quad (B) \frac{m \omega^2 l^2}{e} \quad (C) \frac{m \omega^2 l^2}{e} \quad (D) \text{None of these} \]

**Solution**

When rod rotates the centripetal acceleration of electron comes from electric field \( E = \frac{m \omega^2}{e} \)

Thus, \( \Delta V = -\int \vec{E} \cdot d\vec{r} = -\int_0^l \frac{m \omega^2}{e} dr = \frac{m \omega^2 l^2}{2e} \)
Solution.

Required angle \( \theta \) = \( \frac{\theta_2 - \theta_1}{2} = \frac{88^\circ - 32^\circ}{2} = \frac{56^\circ}{2} = 28^\circ \)

Example #9

The electric potential in a region is given by the relation \( V(x) = 4 + 5x^2 \). If a dipole is placed at position (-1,0) with dipole moment \( \vec{p} \) pointing along positive Y-direction, then

(A) Net force on the dipole is zero.
(B) Net torque on the dipole is zero
(C) Net torque on the dipole is not zero and it is in clockwise direction
(D) Net torque on the dipole is not zero and it is in anticlockwise direction

Solution

\[ V(x) = 4 + 5x^2 \Rightarrow \vec{E} = 10x\hat{i} \]

\[ \therefore \text{Net force will be zero and torque not zero} \]

and rotation will be along clockwise direction

Example #10 to 12

A thin homogeneous rod of mass \( m \) and length \( \ell \) is free to rotate in vertical plane about a horizontal axle pivoted at one end of the rod. A small ball of mass \( m \) and charge \( q \) is attached to the opposite end of this rod. The whole system is positioned in a constant horizontal electric field of magnitude \( E = \frac{mg}{2q} \). The rod is released from shown position from rest.

10. What is the angular acceleration of the rod at the instant of releasing the rod?

(A) \( \frac{8g}{9}\ell \)
(B) \( \frac{3g}{2\ell} \)
(C) \( \frac{9g}{8\ell} \)
(D) \( \frac{2g}{9\ell} \)

11. What is the acceleration of the small ball at the instant of releasing the rod?

(A) \( \frac{8g}{9} \)
(B) \( \frac{9g}{8} \)
(C) \( \frac{7g}{8} \)
(D) \( \frac{8g}{7} \)

12. What is the speed of ball when rod becomes vertical?

(A) \( \sqrt{\frac{2g\ell}{3}} \)
(B) \( \sqrt{\frac{3g\ell}{4}} \)
(C) \( \sqrt{\frac{3g\ell}{2}} \)
(D) \( \sqrt{\frac{4g\ell}{3}} \)

Solution

10. Ans. (C)

By taking torque about hinge \( l_\alpha = mg\left(\frac{\ell}{2}\right) + mg(l) \) when \( l = \frac{m\ell^2}{3} + m\ell^2 \Rightarrow \alpha = \frac{9g}{8\ell} \)
11. Ans. (B)

Acceleration of ball = $a = \left( \frac{9g}{8\ell} \right) \ell = \frac{9}{8} g$

12. Ans. (C)

From work energy theorem

$$\frac{1}{2} \omega^2 = mg \left( \frac{\ell}{2} \right) + mg \ell - qE \ell$$

$$\frac{1}{2} \left( \frac{4}{3} ml^2 \right) \omega^2 = \frac{3}{2} mg \ell - \frac{mg \ell}{2} \Rightarrow \frac{2}{3} ml^2 \omega^2 = mg \ell \Rightarrow \omega = \sqrt{\frac{3g}{2\ell}}$$

Speed of ball = $\omega \ell = \sqrt{\frac{3gl}{2}}$

Example#13

A simple pendulum is suspended in a lift which is going up with an acceleration of 5 m/s$^2$. An electric field of magnitude 5 N/C and directed vertically upward is also present in the lift. The charge of the bob is 1 μC and mass is 1 mg. Taking $g = \pi^2$ and length of the simple pendulum 1m, find the time period of the simple pendulum (in sec).

Solution

$$T = 2\pi \sqrt{\frac{\ell}{g_{eff}}}$$

$$g_{eff} = g - \frac{qE}{M} + 5 = 15 - \frac{1 \times 5 \times 10^{-6}}{1 \times 10^{-6}}$$

$$g_{eff} = 10 = \pi^2$$

$$T = 2 \text{ sec}$$

Example#14

The variation of potential with distance $x$ from a fixed point is shown in figure. Find the magnitude of the electric field (in V/m) at $x = 13$ m.

Solution

$$E = - \frac{dV}{dx} = \frac{20}{4} = 5 \text{ volt/meter}$$

Example#15

The energy density $u$ is plotted against the distance $r$ from the centre of a spherical charge distribution on a log-log scale. Find the magnitude of slope of obtained straight line.

Solution

$$u = \frac{1}{2} \varepsilon_0 E^2 = \frac{1}{2} \varepsilon_0 \left( \frac{q}{4\pi \varepsilon_0 r^2} \right)^2 = \frac{q^2}{32\pi^2 \varepsilon_0 r^2} \Rightarrow \log u = \log \left( \frac{q^2}{32\pi^2 \varepsilon_0 r^2} \right) = \log k - 2 \log r$$
Example #16
The figure shows four situations in which charges as indicated (q>0) are fixed on an axis. How many situations is there a point to the left of the charges where an electron would be in equilibrium?

\[ \begin{array}{c}
+q \\
-4q \\
+4q \\
-4q \\
-4q \\
+q \\
+q \\
+q \\
\end{array} \]

Solution
For (1):
Let the electron be held at a distance \( x \) from +q charge.

For equilibrium \( \frac{q(-e)}{4 \pi \varepsilon_0 x^2} = \frac{(-e)(-4q)}{4 \pi \varepsilon_0 (x + d)^2} \)

We can find value of \( x \) for which \( F_{\text{net}} = 0 \) which means that electron will be in equilibrium.

For (2):
For equilibrium \( \frac{(-e)(-q)}{4 \pi \varepsilon_0 x^2} = \frac{(-e)4q}{4 \pi \varepsilon_0 (x + d)^2} \)

We can find value of \( x \) for which \( F_{\text{net}} = 0 \) which means that electron will be in equilibrium.

In case (3) and (4) the electron will not remain at rest, since it experiences a net non-zero force.

OR
Equilibrium is always found near the smaller charge

Example #17
An electric field is given by \( \vec{E} = (y\hat{i} + x\hat{j}) \frac{N}{C} \). Find the work done (in J) in moving a 1C charge from \( \vec{r}_A = (2\hat{i} + 2\hat{j}) \) m to \( \vec{r}_B = (4\hat{i} + \hat{j}) m \).

Solution
\( A = (2,2) \) and \( B = (4,1) \); \( W_{A\rightarrow B} = q(V_B - V_A) = \int_A^B \vec{E} \cdot d\vec{r} = \int_A^B q\vec{E} \cdot d\vec{r} \)

\( = q \int_A^B [(y\hat{i} + x\hat{j}) \cdot (dx\hat{i} + dy\hat{j})] = q \int_A^B (ydx + xdy) = q \left[ xy \right]_{(2,2)}^{(4,1)} - q \left[ 4 - 4 \right] = 0 \)

Example #18
The arrangement shown consists of three elements.
(i) A thin rod of charge \(-3.0 \mu C\) that forms a full circle of radius 6.0 cm.
(ii) A second thin rod of charge \(2.0 \mu C\) that forms a circular arc of radius 4.0 cm and concentric with the full circle, subtending an angle of 90° at the centre of the full circle.
(iii) An electric dipole with a dipole moment that is perpendicular to a radial line and has magnitude 1.28 \(10^{-21}\) C·m.

Find the net electric potential in volts at the centre.

Solution
Potential due to dipole at the centre of the circle is zero.

Potentials due to charge on circle \( V_1 = \frac{K(-3 \times 10^{-6})}{6 \times 10^{-2}} \)

Potential due to arc \( V_2 = \frac{K(2 \times 10^{-6})}{4 \times 10^{-2}} \) Net potential = \( V_1 + V_2 = 0 \)
Example #19

Six charges are kept at the vertices of a regular hexagon as shown in the figure. If magnitude of force applied by +Q on +q charge is F, then net electric force on the +Q is nF. Find the value of n.

Solution

\[ F_{net} = 9F \]

Example #20

Electric field in a region is given by \( \mathbf{E} = -4x\mathbf{i} + 6y\mathbf{j} \). The charge enclosed in the cube of side 1m oriented as shown in the diagram is given by \( \alpha \epsilon_0 \). Find the value of \( \alpha \).

Solution

\[ \phi = (6y) \text{Area} - (4x) \text{Area} = 6 \cdot 1 - 4 \cdot 1^2 = 2 \text{ therefore } \frac{q}{\epsilon_0} = 2 \Rightarrow q = 2\epsilon_0 \]

Example #21

An infinite plane of charge with \( \sigma = 2 \epsilon_0 \frac{C}{m^2} \) is tilted at a 37° angle to the vertical direction as shown below. Find the potential difference, \( V_A - V_B \) in volts, between points A and B at 5 m distance apart. (where B is vertically above A).

Solution

\[ E = \frac{\sigma}{2\epsilon_0} = 1 \text{ N/C} \quad \therefore V_B - V_A = -\int \mathbf{E} \cdot d\mathbf{r} = \left( \frac{\sigma}{2\epsilon_0} \right) (5 \cos 53°) = 3V \]