Polynomials

- **Polynomial**
  An algebraic expression in which the exponents of the variables are non-negative integers are called polynomials. For example, $3x^4 + 2x^3 + x + 9$, $3x^4$ etc are polynomials.

- **Constant polynomial:** A constant polynomial is of the form $p(x) = k$, where k is a real number. For example, –9, 10, 0 are constant polynomials.

- **Zero polynomial:** A constant polynomial '0' is called zero polynomial.

  General form of a polynomial:

  A polynomial of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_0, a_1 \ldots a_r$ are constants and $a_n \neq 0$.
  Here, $a_0, a_1 \ldots a_n$ are the respective coefficients of $x^0, x^1, x^2, \ldots x^n$ and n is the power of the variable x.
  $a_n x^n, a_{n-1} x^{n-1} - a_0$ and $a_0 \neq 0$ are called the terms of $p(x)$.

- **Classification of polynomials on the basis of number of terms**

  - A polynomial having one term is called a monomial e.g. $3x$, $25t^3$ etc.
  - A polynomial having two terms is called a binomial e.g. $2t - 6$, $3x^4 + 2x$ etc.
  - A polynomial having three terms is called a trinomial. e.g. $3x^4 + 8x + 7$ etc.

- **Degree**

  The degree of a polynomial is the highest exponent of the variable of the polynomial. For example, the degree of polynomial $3x^4 + 2x^3 + x + 9$ is 4.
  The degree of a term of a polynomial is the value of the exponent of the term.

- **Classification of polynomial according to their degrees**

- A polynomial of degree one is called a linear polynomial e.g. $3x + 2$, $4x$, $x + 9$.
- A polynomial of degree two is called a quadratic polynomial. e.g. $x^2 + 9$, $3x^2 + 4x + 6$.
- A polynomial of degree three is called a cubic polynomial e.g. $10x^3 + 3$, $9x^3$.

  Note: The degree of a non-zero constant polynomial is zero and the degree of a zero polynomial is not defined.

- **Values of polynomials at different points**

A polynomial is made up of constants and variables. Hence, the value of the polynomial changes as the value of the variable in the polynomial changes. Thus, for the different values of the variable *x*, we get different values of the polynomial.

Example:

Find the value of polynomial $p(x) = 3x^2 + 2x + 9$ at $x = -2$.

Solution:

The variable in the given polynomial is *x*. Hence, replacing *x* by −2.

$p\ (x) = 3x^3 + 2x + 9$

$\therefore p\ (-2) = 3\ (-2)^3 + 2\ (-2) + 9$

$= -24 - 4 + 9$

$= -19$

- **Zeroes of a polynomial**

A real number *a* is said to be the zero of polynomial $p(x)$ if $p(a) = 0$. In this case, *a* is also called the root of the equation $p(x) = 0$.

Note:

- The maximum number of roots of a polynomial is less than or equal to the degree of the polynomial.
- A non-zero constant polynomial has no zeroes.
- A polynomial can have more than one zero.

Example: Check if $-\frac{3}{2}$ is a zero of polynomial, $p(x) = 2x^2 - 7x - 15$.

Solution:

Put $x = -\frac{3}{2}$ in the given polynomial $p(x)$.

$$\therefore p\left(-\frac{3}{2}\right) = 2\left(-\frac{3}{2}\right)^2 - 7\left(-\frac{3}{2}\right) - 15$$

$$= 2\times\frac{9}{4} + \frac{21}{2} - 15$$

$$= \frac{9}{2} + \frac{21}{2} - 15$$

$$= \frac{9+21-30}{2}$$

$$= 0$$

Hence, $-\frac{3}{2}$ is a zero of polynomial $p(x)$.

- **Division of a polynomial by a monomial using long division method**

Example: Divide $x^4 - 2x^3 - 2x^2 + 7x - 15$ by $x - 2$.

Solution:

$$
\begin{array}{r|l}
 & x^3 - 2x + 3 \\
\hline
x - 2 & x^4 - 2x^3 - 2x \ + 7x - 15 \\
 & x^4 - 2x^3 \\
 & - \quad + \\
\hline
 & \qquad - 2x \ + 7x - 15 \\
 & \qquad -2x^2 + 4x \\
 & \qquad + \quad - \\
\hline
 & \qquad\qquad 3x - 15 \\
 & \qquad\qquad 3x - 6 \\
 & \qquad\qquad - \quad + \\
\hline
 & \qquad\qquad\quad -9
\end{array}
$$

Divison of polynomials by monomials also satisfy Division algorithm i.e., **Dividend = Divisor × Quotient + Remainder**

It can be easily verified that here $(x^4 - 2x^3 - 2x^2 + 7x - 15) = (x - 2)(x^3 - 2x + 3) + (-9)$.

- **Remainder Theorem**

If $p(x)$ is a polynomial of degree greater than or equal to one and a is any real number then if $p(x)$ is divided by the linear polynomial $x - a$, the remainder is $p(a)$.

Example: Find the remainder when $x^5 - x^2 + 5$ is divided by $x - 2$.

Solution: $p(x) = x^5 - x^2 + 5$
The zero of $x - 2$ is 2.
$p(2) = 2^5 - 2^2 + 5 = 32 - 4 + 5 = 33$
Therefore, by remainder theorem, the remainder is 33.

- **Factor Theorem**

If $p(x)$ is a polynomial of degree $n \geq 1$ and a is any real number, then

- $x - a$ is a factor of $p(x)$, if $p(a) = 0$.
- $p(a) = 0$, if $(x - a)$ is a factor of $p(x)$.

Example: Determine whether $x + 3$ is a factor of $x^3 + 5x^2 + 5x - 3$.

Solution: The zero of $x + 3$ is -3.
Let $p(x) = x^3 + 5x^2 + 5x - 3$

$p(-3) = (-3)^3 + 5(-3)^2 + 5(-3) - 3$
$= -27 + 45 - 15 - 3$
$= -45 + 45$
$= 0$
Therefore, by factor theorem, $x + 3$ is the factor of $p(x)$.

- Factorisation of quadratic polynomials of the form $ax^2 + bx + c$ can be done using Factor theorem and splitting the middle term.

Example 1: **Factorize $x^2 - 7x + 10$ using the factor theorem.**

Solution: Let $p(x) = x^2 - 7x + 10$
The constant term is 10 and its factors are ±1, ±2, ±5and ±10.

Let us check the value of the polynomial for each of these factors of 10.

$p(1) = 1^2 - 7 \times 1 + 10 = 1 - 7 + 10 = 4 \neq 0$
Hence, $x - 1$ is not a factor of $p(x)$.

$p(2) = 2^2 - 7 \times 2 + 10 = 4 - 14 + 10 = 0$
Hence, $x - 2$ is a factor of $p(x)$.

$p(5) = 5^2 - 7 \times 5 + 10 = 25 - 35 + 10 = 0$
Hence, $x - 5$ is a factor of $p(x)$.

We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $x - 2$ and $x - 5$.
Thus, we can write the given polynomial as:

$p(x) = x^2 - 7x + 10 = (x - 2)(x - 5)$

Example 2: Factorize $2x^2 - 11x + 15$ by splitting the middle term.

Solution: The given polynomial is $2x^2 - 11x + 15$.
Here, $a\,c = 2 \times 15 = 30$. The middle term is − 11. Therefore, we have to split − 11 into two numbers such that their product is 30 and their sum is −11. These numbers are − 5 and − 6 [As $(-5) + (-6) = -11$ and $(-5) \times (-6) = 30$].

Thus, we have:

$2x^2 - 11x + 15 = 2x^2 - 5x - 6x + 15$
$= x(2x - 5) - 3(2x - 5)$
$= (2x - 5)(x - 3)$

Note: A quadratic polynomial can have a maximum of two factors.

- Factorisation of cubic polynomials of the form $ax^3 + bx^2 + cx + d$ can be done using factor theorem and hit and trial method.

  A cubic polynomial can have a maximum of three linear factors. So, by knowing one of these factors, we can reduce it to a quadratic polynomial.
  Thus, to factorize a cubic polynomial, we first find a factor by the hit and trial method or by using the factor theorem, and then reduce the cubic polynomial into a quadratic polynomial and it is then solved further.

  Example: Factorise $p(x) = x^3 - 7x + 6$

  Solution: The constant term is 6.
  The factors of 6 are ±1, ±2, ±3 and ±6.
  Let $x = 1$
  $p\ (x = 1) = (1)^3 - 7\ (1) + 6$
  $= 1 - 7 + 6$
  $= 0$
  Thus, $(x - 1)$ is a factor of $p(x)$.
  Now we have to group the term of $p\ (x)$ such that we can take $(x - 1)$ as common.
  Therefore, $p(x) = x^3 - 7x + 6$
  $= x^3 - x^2 + x^2 - x - 6x + 6$
  $= x^2\ (x - 1) + x\ (x - 1) - 6\ (x - 1)$
  $= (x - 1)\ (x^2 + x - 6)$ ... (1)
  Now, we factorize $(x^2 + x - 6)$ by splitting its middle term.

  $x^2 + x - 6 = x^2 + 3x - 2x - 6$
  $= x\ (x + 3) - 2\ (x + 3)$
  $= (x - 2)\ (x + 3)$
  From equation (1), we get
  $p(x) = (x - 1)\ (x - 2)\ (x + 3)$

  Hence, factors of polynomial $p(x)$ are $(x - 1)$, $(x - 2)$ and $(x + 3)$.

- **Identity:** $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

  We can use this identity to factorize and expand the polynomials.
  For example, the given expression can be factorized as follows:
  $2x^2 + 27y^2 + 25z^2 + 6\sqrt{6}xy - 30\sqrt{3}yz - 10\sqrt{2}xz$
  $= \left(\sqrt{2}x\right)^2 + \left(3\sqrt{3}y\right)^2 + \left(-5z\right)^2 + 2\cdot\left(\sqrt{2}x\right)\left(3\sqrt{3}y\right) + 2\left(3\sqrt{3}y\right)(-5z) + 2\left(\sqrt{2}x\right)(-5z)$
  On comparing the expression with $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$, we get

$$2x^2 + 27y^2 + 25z^2 + 6\sqrt{6}xy - 30\sqrt{3}yz - 10\sqrt{2}xz$$
$$= \left(\sqrt{2}x + 3\sqrt{3}y - 5z\right)^2$$

- **Identities:** $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$ and $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$

  Other ways to represent these identities are:

  - $x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$
  - $x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$
  - $x^3 - y^3 = (x - y)^3 + 3xy\,(x - y)$
  - $x^3 - y^3 = (x - y)\,(x^2 + xy + y^2)$

Example: Expand $(3x + 2y)^3 - (3x - 2y)^3$

Solution: $(3x + 2y)^3 = (3x)^3 + (2y)^3 + 3\,(3x)\,(2y)\,(3x + 2y)$
$= 27x^3 + 8y^3 + 54x^2y + 36xy^2$ ... (1)
$(3x - 2y)^3 = (3x)^3 - (2y)^3 - 3\,(3x)\,(2y)\,(3x - 2y)$
$= 27x^3 - 8y^3 - 54x^2y + 36xy^2$ ... (2)
From equations (1) and (2) in given expression, we get
$(3x + 2y)^3 - (3x - 2y)^3 = (27x^3 + 8y^3 + 54x^2y + 36xy^2) - (27x^3 - 8y^3 - 54x^2y + 36xy^2)$
$= 27x^3 + 8y^3 + 54x^2y + 36xy^2 - 27x^3 + 8y^3 + 54x^2y - 36xy^2$
$= 16y^3 + 108x^2y$

Identity: $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

The expression $2\sqrt{2}x^3 + 3\sqrt{3}y^3 + 6\sqrt{6}z^3 - 18xyz$ can be factorized using this identity as follows:

$$2\sqrt{2}x^3 + 3\sqrt{3}y^3 + 6\sqrt{6}z^3 - 18xyz$$

$$= \left(\sqrt{2}x\right)^3 + \left(\sqrt{3}y\right)^3 + \left(\sqrt{6}z\right)^3 - 3\left(\sqrt{2x}\right)\left(\sqrt{3}y\right)\left(\sqrt{6}z\right)$$

$$= \left(\sqrt{2}x + \sqrt{3}y + \sqrt{6}z\right)\left[\left(\sqrt{2}x\right)^2 + \left(\sqrt{3}y\right)^2 + \left(\sqrt{6}z\right)^2 - \left(\sqrt{2}x\right)\left(\sqrt{3}y\right) - \left(\sqrt{3}y\right)\left(\sqrt{6}z\right) - \left(\sqrt{6}z\right)\left(\sqrt{2}x\right)\right]$$

$$= \left(\sqrt{2}x + \sqrt{3}y + \sqrt{6}z\right)\left(2x^2 + 3y^2 + 6z^2 - \sqrt{6}xy - 3\sqrt{2}yz - 2\sqrt{3}zx\right)$$

Special case: If $(x + y + z) = 0$ then $x^3 + y^3 + z^3 = 3xyz$.