

Quadrilaterals

A Diagonal of a Parallelogram Divides It into Two Congruent Triangles

The Diagonal of a Parallelogram Divides It into Two Congruent Triangles

Look at the two triangular sandwiches obtained by cutting a sandwich along the **diagonal**.



Are the two divided parts of the sandwich equal in **area**? What can we say about their congruency? Does the diagonal divide the sandwich into two triangles of equal area?

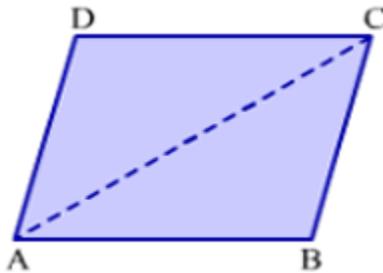
The answers to these questions are based on a property of **parallelograms**. It can be stated as follows:

A diagonal of a parallelogram divides it into two congruent triangles.

In this lesson, we will understand and prove the above-stated property of parallelograms. We will also solve some examples related to the same.

The Diagonal of a Parallelogram Divides It into Two Congruent Triangles

Consider a parallelogram ABCD. AC is a diagonal of this parallelogram.



Suppose the area of $\triangle ABC$ is 15 cm^2 .

We can find the area of the given parallelogram with the help of the area of ΔABC by using a property of parallelograms which states that:

A diagonal of a parallelogram divides it into two congruent triangles.

In the given figure, diagonal AC divides the parallelogram into two congruent triangles, ΔABC and ΔCDA .

We know that congruent figures are equal in area.

So, $\text{ar}(\Delta ABC) = \text{ar}(\Delta CDA)$

$\therefore \text{ar}(\text{parallelogram } ABCD) = 2 \times \text{ar}(\Delta ABC)$

$= 2 \times 15 \text{ cm}^2$

$= 30 \text{ cm}^2$

Concept Builder

- A trapezium is a quadrilateral having one pair of parallel opposite sides.
- A rectangle is a quadrilateral having two pairs of equal opposite sides. Also, each angle in a rectangle is equal to 90° .
- A rhombus is a quadrilateral having all sides equal.
- A square is a quadrilateral having all sides equal and all angles equal to 90° .

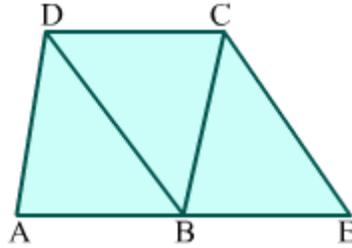
Did You Know?

- The quadrilateral formed by joining the midpoints of a quadrilateral is a parallelogram.
- If the diagonals of a quadrilateral are perpendicular to each other, then the quadrilateral formed by joining the midpoints of the quadrilateral is a rectangle.
- The sum of any three sides of a quadrilateral is always greater than the fourth side.
- The sum of all sides of a quadrilateral is greater than the sum of its diagonals.

Solved Examples

Easy

Example 1: In the given figure, ABCD and BECD are two parallelograms. Prove that $\Delta ABD \cong \Delta BEC$.



Solution:

ABCD is a parallelogram with BD as a diagonal and BECD is a parallelogram with BC as a diagonal. We know that a diagonal of a parallelogram divides it into two congruent triangles.

$$\therefore \triangle ABD \cong \triangle CDB$$

$$\text{And } \triangle CDB \cong \triangle BEC$$

$$\Rightarrow \triangle ABD \cong \triangle BEC$$

Example 2: ABCD is a parallelogram with area 60 cm². Find the area of $\triangle ACD$.

Solution:

We know that a diagonal of a parallelogram divides it into two congruent triangles.

Also, congruent figures are equal in area.

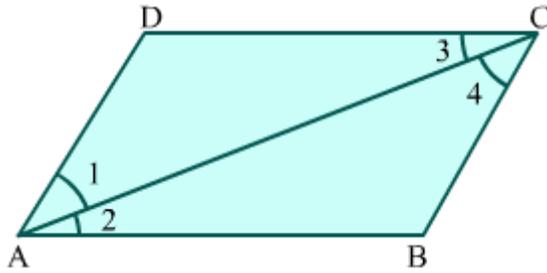
So, we have:

$$\text{ar} (\triangle ACD) = \text{ar} (\triangle CAB) = \frac{\text{ar} (\text{parallelogram ABCD})}{2}$$

$$\Rightarrow \text{ar} (\triangle ACD) = \frac{60 \text{ cm}^2}{2}$$

$$\Rightarrow \therefore \text{ar} (\triangle ACD) = 30 \text{ cm}^2$$

Example 3: ABCD is a parallelogram in which $\angle 1 = \angle 2$. Prove that $\angle 3 = \angle 4$.



Solution:

ABCD is a parallelogram with AC as a diagonal. We know that a diagonal of a parallelogram divides it into two congruent triangles.

$$\therefore \triangle ABC \cong \triangle CDA$$

$$\Rightarrow \angle 2 = \angle 3 \text{ and } \angle 1 = \angle 4 \dots (1)$$

It is given that $\angle 1 = \angle 2$ (2)

From 1 and 2, we get:

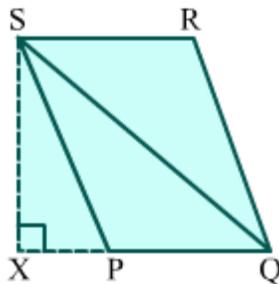
$$\angle 3 = \angle 4$$

Medium

Example 1: The area of a parallelogram PQRS is 50 cm^2 . Find the distance between PQ and SR if the length of PQ is 6 cm.

Solution:

Construction: Draw diagonal SQ of the given parallelogram PQRS. Extend PQ and draw a line perpendicular to it from point S.



We know that a diagonal of a parallelogram divides it into two congruent triangles.

Also, congruent figures are equal in area.

$$\therefore \text{ar} (\Delta PQS) = \text{ar} (\Delta RSQ)$$

$$\text{Now, ar (parallelogram PQRS)} = \text{ar} (\Delta PQS) + \text{ar} (\Delta RSQ)$$

$$\Rightarrow \text{ar (parallelogram PQRS)} = 2 \times \text{ar} (\Delta PQS)$$

$$\Rightarrow \text{ar} (\Delta PQS) = \frac{1}{2} \times \text{ar (parallelogram PQRS)}$$

$$\Rightarrow \text{ar} (\Delta PQS) = \frac{50}{2} \text{ cm}^2$$

$$\Rightarrow \therefore \text{ar} (\Delta PQS) = 25 \text{ cm}^2$$

$$\text{Also, ar} (\Delta PQS) = \frac{1}{2} \times PQ \times SX$$

$$\Rightarrow \frac{1}{2} \times PQ \times SX = 25 \text{ cm}^2$$

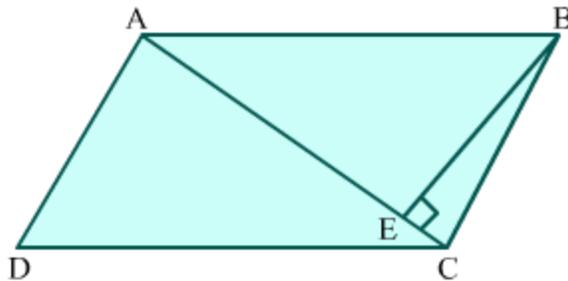
$$\Rightarrow PQ \times SX = 50 \text{ cm}^2$$

$$\Rightarrow SX = \frac{50}{6} \text{ cm} \quad (\because \text{It is given that } PQ = 6 \text{ cm})$$

$$\Rightarrow \therefore SX = 8.3 \text{ cm}$$

Thus, the distance between PQ and SR is 8.3 cm.

Example 2: In the given parallelogram ABCD, altitude BE on AC is of length 4 cm. If the length of diagonal AC is 6 cm, then find the area of the parallelogram.



Solution:

We know that a diagonal of a parallelogram divides it into two congruent triangles.

Also, congruent figures are equal in area.

$$\therefore \text{ar}(\triangle ABC) = \text{ar}(\triangle CDA)$$

$$\text{Now, ar}(\triangle ABC) = \frac{1}{2} \times \text{Base} \times \text{Height}$$

$$= \frac{1}{2} \times AC \times BE$$

$$= \frac{1}{2} \times 6 \text{ cm} \times 4 \text{ cm}$$

$$= 12 \text{ cm}^2$$

$$\text{Therefore, ar}(\text{parallelogram ABCD}) = \text{ar}(\triangle ABC) + \text{ar}(\triangle CDA)$$

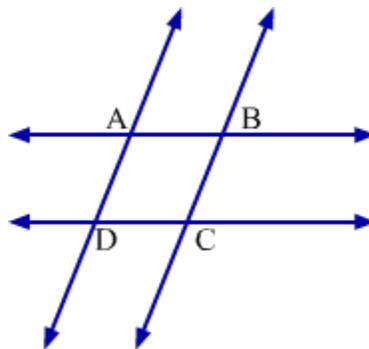
$$= 2 \times \text{ar}(\triangle ABC)$$

$$= 2 \times 12 \text{ cm}^2$$

$$= 24 \text{ cm}^2$$

Properties of The Sides of a Parallelogram**Property of the Sides of a Parallelogram**

Consider the given pairs of parallel lines.



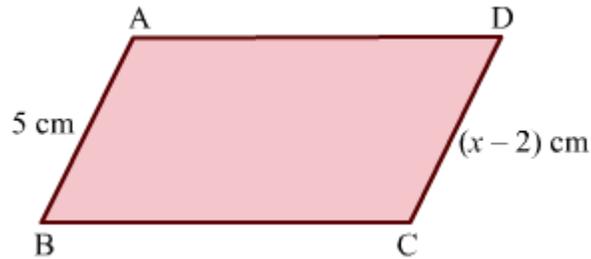
A closed figure ABCD is formed by the intersection of the two pairs of parallel lines. This figure is a parallelogram. A property of parallelograms defines the relation between the sides of a parallelogram as follows:

Opposite sides of a parallelogram are equal.

In this lesson, we will study the above-stated property and its converse. We will also solve some examples related to the same.

Opposite Sides of a Parallelogram Are Equal

Consider the given parallelogram ABCD.



We can find the value of x by using the property of parallelograms which states that:

Opposite sides of a parallelogram are equal.

Thus, in the given figure, we have $AB = DC$ and $AD = BC$.

Since $AB = DC$, we have:

$$x - 2 = 5$$

$$\Rightarrow x = 7$$

Concept Builder

- A quadrilateral is a polygon having four sides.
- The sum of the interior angles of a quadrilateral is 360° .

Know More

- A pentadecagon is a fifteen-sided polygon. The sum of its interior angles is 2340° .
- An icosagon is a twenty-sided polygon. The sum of its interior angles is 3240° .

Did You Know?

The headquarters of the US Department of Defense is called 'the Pentagon'. It is one of the world's largest office buildings. It is virtually a city in itself.

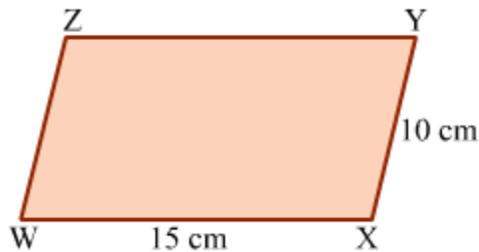


Solved Examples

Easy

Example 1: What is the perimeter of the given parallelogram WXYZ if $WX = 15$ cm and $XY = 10$ cm?

Solution:



We know that the opposite sides of a parallelogram are equal.

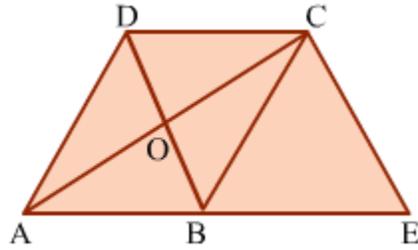
$$\therefore WX = ZY = 15 \text{ cm and } XY = WZ = 10 \text{ cm}$$

$$\begin{aligned} \text{Perimeter of parallelogram WXYZ} &= WX + XY + YZ + ZW \\ &= (15 + 10 + 15 + 10) \text{ cm} \end{aligned}$$

$$= 50 \text{ cm}$$

Medium

Example 1: In the given figure, ABCD is a parallelogram and B is the midpoint of AE. If $DB = CE$, then prove that BECD is also a parallelogram.



Solution:

We know that the opposite sides of a parallelogram are equal.

$$\therefore AB = DC \dots (1)$$

It is given that B is the midpoint of AE.

$$\therefore AB = BE \dots (2)$$

From equations 1 and 2, we get:

$$DC = BE$$

Also, it is given that $DB = CE$.

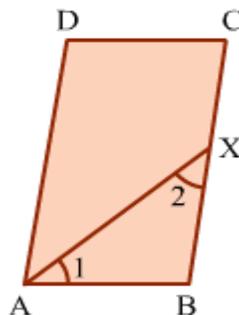
Now, in quadrilateral BECD, the opposite sides are equal (i.e., $DC = BE$ and $DB = CE$). Therefore, it is a parallelogram.

Hard

Example 1: In a parallelogram ABCD, the bisector of $\angle BAD$ also bisects side BC. Prove that the length of side AD is twice the length of side AB.

Solution:

The parallelogram ABCD according to the given specifications is shown below.



Here, AX is the bisector of $\angle BAD$.

$$\therefore \angle 1 = \frac{1}{2} \angle BAD \quad \dots(1)$$

Since ABCD is a parallelogram, $AD \parallel BC$ and AB is the transversal between these lines.

$$\therefore \angle BAD + \angle CBA = 180^\circ \dots (2)$$

In $\triangle ABX$, by the angle sum property of triangles, we have:

$$\angle 1 + \angle 2 + \angle ABX = 180^\circ$$

$$\Rightarrow \frac{1}{2} \angle BAD + \angle 2 + 180^\circ - \angle BAD = 180^\circ \quad (\text{Using equations 1 and 2})$$

$$\Rightarrow \angle 2 + \frac{1}{2} \angle BAD = 0$$

$$\Rightarrow \angle 2 = \frac{1}{2} \angle BAD$$

$$\Rightarrow \angle 2 = \angle 1$$

We know that the sides opposite equal angles are also equal.

$$\therefore AB = BX \dots (3)$$

Since ABCD is a parallelogram, $AD = BC$.

$$\text{Now, } BC = BX + XC$$

$$\Rightarrow AD = BX + XC$$

$$\Rightarrow AD = 2BX (\because AX \text{ bisects } BC)$$

$$\Rightarrow \therefore AD = 2AB \text{ (Using equation 3)}$$

Thus, in parallelogram ABCD, the length of side AD is twice the length of side AB.

Properties of The Angles of a Parallelogram

Look at the postage stamp shown below.



Observe how the stamp is shaped like a parallelogram. What can you say about its opposite angles? Is there any relation between them? Are they equal?

A property of parallelograms relates the opposite angles of a parallelogram as follows:

Opposite angles of a parallelogram are equal.

In this lesson, we will study the above-stated property and its converse. We will also solve some examples based on the same.

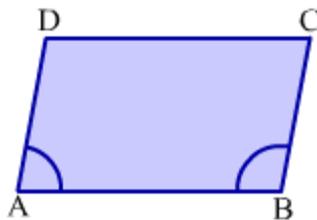
Whiz Kid

The sum of the measures of all the exterior angles of a quadrilateral (i.e., one at each vertex) is equal to the sum of the measures of all the interior angles of the quadrilateral, i.e., 360° .

Concept Builder

Adjacent angles in a parallelogram are supplementary.

In parallelogram ABCD, $AD \parallel BC$ and AB is the transversal intersecting these lines.



Therefore, $\angle A$ and $\angle B$ are **interior angles**

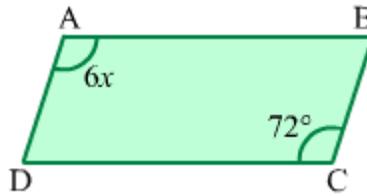
on the same side of the transversal and, hence, **supplementary**

Similarly, we can say that $\angle B$ and $\angle C$, $\angle C$ and $\angle D$, and $\angle D$ and $\angle A$ are supplementary angles.

Solved Examples

Easy

Example 1: Find the value of x if ABCD is a parallelogram.



Solution:

We know that the opposite angles of a parallelogram are equal.

$$\therefore \angle A = \angle C$$

$$\Rightarrow 6x = 72^\circ$$

$$\therefore x = 12^\circ$$

Example 2: Find the measure of all the angles of a parallelogram whose adjacent angles are in the ratio 1:2.

Solution:

In a parallelogram ABCD, let $\angle A = x^\circ$ and $\angle B = 2x^\circ$.

In a parallelogram, the adjacent angles are supplementary.

$$\angle A + \angle B = 180^\circ$$

$$\Rightarrow x^\circ + 2x^\circ = 180^\circ$$

$$\Rightarrow 3x^\circ = 180$$

$$\Rightarrow x^\circ = \frac{180^\circ}{3}$$

$$\Rightarrow x^\circ = 60^\circ$$

Thus, we get

$$\angle A = x^\circ = 60^\circ$$

$$\angle B = 2x^\circ = 2 \times 60^\circ = 120^\circ$$

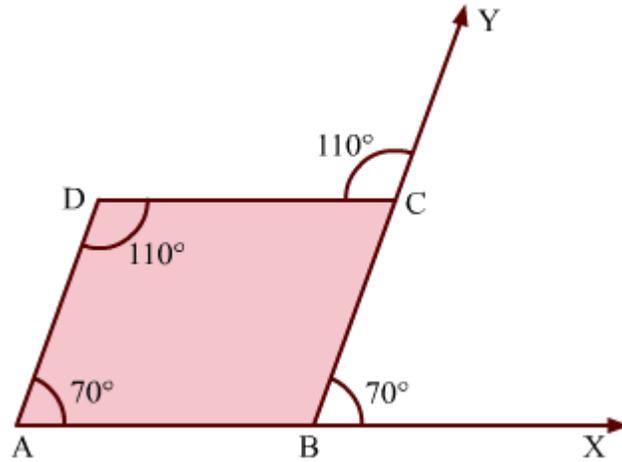
In a parallelogram, the opposite angles are equal.

Thus, we get

$$\angle A = \angle C = 60^\circ \text{ and } \angle B = \angle D = 120^\circ$$

Medium

Example 1: Is the shown quadrilateral ABCD a parallelogram?



Solution:

In the given figure, $\angle CBX$ and $\angle CBA$ form a linear pair.

$$\therefore \angle CBX + \angle CBA = 180^\circ$$

$$\Rightarrow \angle CBA = 180^\circ - \angle CBX$$

$$\Rightarrow \angle CBA = 180^\circ - 70^\circ$$

$$\Rightarrow \angle CBA = 110^\circ$$

$$\Rightarrow \angle CBA = \angle CDA$$

Similarly, $\angle DCY$ and $\angle BCD$ form a linear pair.

$$\therefore \angle DCY + \angle BCD = 180^\circ$$

$$\Rightarrow \angle BCD = 180^\circ - \angle DCY$$

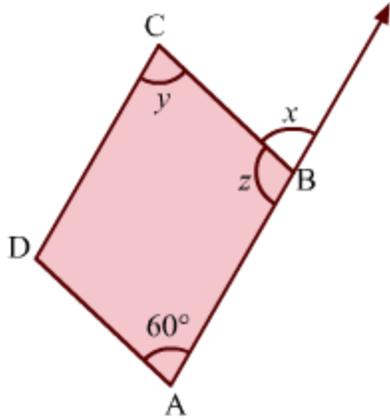
$$\Rightarrow \angle BCD = 180^\circ - 110^\circ$$

$$\Rightarrow \angle BCD = 70^\circ$$

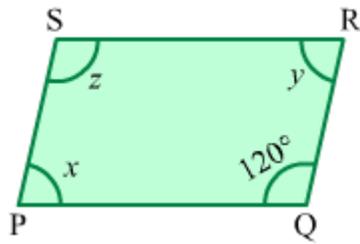
$$\Rightarrow \angle BCD = \angle BAD$$

Thus, quadrilateral ABCD has two pairs of equal opposite angles. Hence, it is a parallelogram.

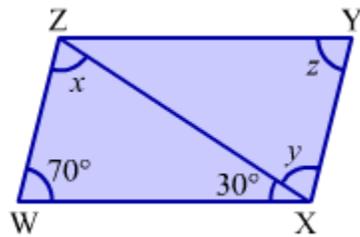
Example 2: Find the values of x, y and z in the following parallelograms.



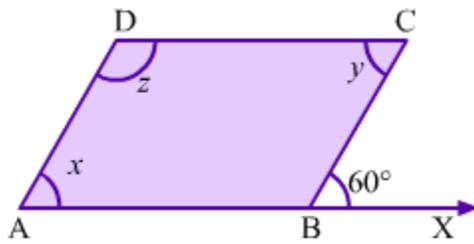
1.



2.



3.



4.

Solution:

1. We know that the opposite angles of a parallelogram are equal.

So, $\angle BCD = \angle DAB$

$\therefore y = 60^\circ$

We also know that the adjacent angles of a parallelogram are supplementary.

$$\text{So, } \angle CBA + \angle DAB = 180^\circ$$

$$\Rightarrow z + 60^\circ = 180^\circ$$

$$\Rightarrow z = 180^\circ - 60^\circ$$

$$\Rightarrow \therefore z = 120^\circ$$

Now, x and z form a linear pair of angles; so, their sum is 180° .

$$\text{So, } x + z = 180^\circ$$

$$\Rightarrow x + 120^\circ = 180^\circ$$

$$\Rightarrow \therefore x = 180^\circ - 120^\circ = 60^\circ$$

2. We know that the opposite angles of a parallelogram are equal.

$$\text{So, } \angle PSR = \angle PQR$$

$$\therefore z = 120^\circ$$

$\angle QPS$ and $\angle PQR$ are adjacent angles.

$$\text{So, } \angle QPS + \angle PQR = 180^\circ$$

$$\boxed{x + 120^\circ = 180^\circ}$$

$$\boxed{x = 180^\circ - 120^\circ}$$

$$\boxed{\boxed{x = 60^\circ}}$$

$\angle QRS$ and $\angle QPS$ are opposite angles.

$$\text{So, } \angle QRS = \angle QPS$$

$$\boxed{y = x}$$

$$\boxed{\therefore y = 60^\circ}$$

3. We know that the opposite angles of a parallelogram are equal.

So, $\angle XYZ = \angle XWZ$

$$\square z = 70^\circ$$

$\angle XYZ$ and $\angle WXY$ are adjacent angles.

$$\square \angle XYZ + \angle WXY = 180^\circ$$

$$\square z + 30^\circ + y = 180^\circ$$

$$\square 70^\circ + 30^\circ + y = 180^\circ$$

$$\square 100^\circ + y = 180^\circ$$

$$\square y = 180^\circ - 100^\circ$$

$$\square \therefore y = 80^\circ$$

Now, $XY \parallel WZ$; so, $\angle WZX$ and $\angle YXZ$ are alternate interior angles.

So, $\angle WZX = \angle YXZ$

$$\square x = y$$

$$\square \therefore x = 80^\circ$$

4. It is given that $\angle CBX = 60^\circ$.

$\angle CBA$ and $\angle CBX$ form a linear pair.

So, $\angle CBA + \angle CBX = 180^\circ$

$$\square \angle CBA + 60^\circ = 180^\circ$$

$$\square \angle CBA = 180^\circ - 60^\circ$$

$$\square \therefore \angle CBA = 120^\circ$$

$\angle CDA$ and $\angle CBA$ are opposite angles.

So, $\angle CDA = \angle CBA$

$$\square z = \angle CBA$$

$$\angle z = 120^\circ$$

$\angle BCD$ and $\angle CBA$ are adjacent angles.

$$\text{So, } \angle BCD + \angle CBA = 180^\circ$$

$$\angle y + 120^\circ = 180^\circ$$

$$\angle y = 180^\circ - 120^\circ$$

$$\therefore y = 60^\circ$$

$\angle BAD$ and $\angle BCD$ are opposite angles.

$$\text{So, } \angle BAD = \angle BCD$$

$$\angle x = y$$

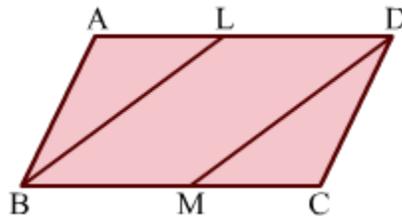
$$\therefore x = 60^\circ$$

Hard

Example 1: Show that the bisectors of opposite angles of a parallelogram are parallel to each other.

Solution:

Let ABCD be a parallelogram. Let BL and DM be the bisectors of $\angle ABC$ and $\angle ADC$ respectively.



Since BL and DM are the bisectors of $\angle ABC$ and $\angle ADC$ respectively, we have:

$$\angle LBM = \frac{\angle ABC}{2} \quad \dots(1)$$

$$\angle LDM = \frac{\angle ADC}{2} \quad \dots(2)$$

We know that the opposite angles of a parallelogram are equal.

$$\therefore \angle ABC = \angle ADC$$

On dividing both sides of the above equation by 2, we obtain:

$$\frac{\angle ABC}{2} = \frac{\angle ADC}{2}$$

Using equations 1 and 2, we obtain:

$$\angle LBM = \angle LDM$$

Now, LD and BM are parallel.

So, $\angle DLB + \angle LBM = 180^\circ$ (Interior angles on the same side of a transversal)

$$\square \angle DLB = 180^\circ - \angle LBM$$

Similarly, $\angle DMB = 180^\circ - \angle LDM$

$$\therefore \angle DLB = \angle DMB (\because \angle LBM = \angle LDM)$$

In quadrilateral LDMB, the opposite angles $\angle DLB$ and $\angle DMB$ are equal. Hence, it is a parallelogram.

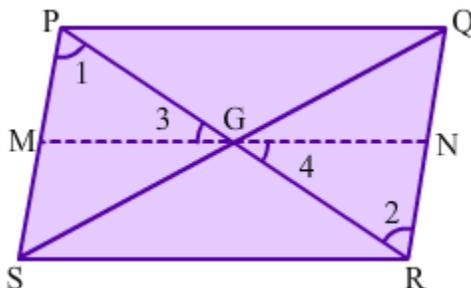
$$\Rightarrow BL \parallel DM$$

We know that BL and DM are the bisectors of opposite angles of parallelogram ABCD. Thus, the bisectors of opposite angles of a parallelogram are parallel.

Properties of The Diagonals of a Parallelogram

Relation between the Diagonals of a Parallelogram

Consider the following parallelogram PQRS.



In the figure, $GM = GN$, but can we prove this?

In order to prove $GM = GN$, we need to show that $\triangle GMP$ is congruent to $\triangle GNR$.

In $\triangle GMP$ and $\triangle GNR$, we have two sets of equal angles as follows:

$\angle 3 = \angle 4$ (Vertically opposite angles)

$\angle 1 = \angle 2$ (Alternate interior angles; since $PS \parallel QR$ and PR is the transversal)

Now, to apply the ASA congruence rule, we need to show that GP and GR are equal.

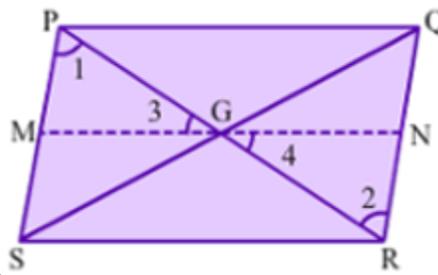
A property of parallelograms helps us establish this equality and it can be stated as follows:

The diagonals of a parallelogram bisect each other.

In this lesson, we will study the above-stated property and solve some problems based on it.

Using the Property

Let us use the property of the diagonals of a parallelogram to solve the problem discussed at the beginning.



Let us once again consider parallelogram PQRS.

We have to prove that $GM = GN$.

Since diagonals PR and QS bisect each other, we obtain:

$GP = GR$ and $GS = GQ$... (1)

In $\triangle GMP$ and $\triangle GNR$, we have:

$\angle 3 = \angle 4$ (Vertically opposite angles)

$\angle 1 = \angle 2$ (Alternate interior angles; since $PS \parallel QR$ and PR is the transversal)

$$GP = GR(\text{Using 1})$$

Thus, by the ASA congruence rule, we obtain:

$$\triangle GMP \cong \triangle GNR$$

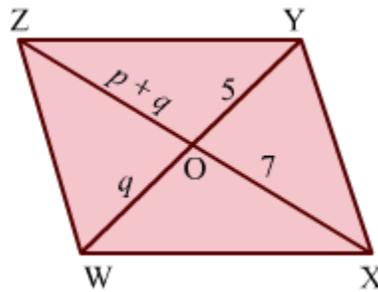
$$\Rightarrow GM = GN(\text{By CPCT})$$

Similarly, we can use the property of the diagonals of a parallelogram to solve other problems.

Solved Examples

Easy

Example 1: If the shown quadrilateral WXYZ is a parallelogram, then find the values of p and q .



Solution:

We know that the diagonals of a parallelogram bisect each other.

$$\therefore WO = OY$$

$$\Rightarrow q = 5$$

Similarly, $XO = OZ$

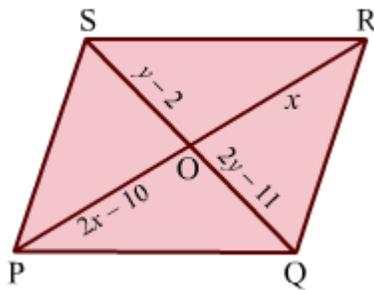
$$\therefore p + q = 7$$

$$\Rightarrow p = 7 - q$$

$$\Rightarrow p = 7 - 5$$

$$\Rightarrow \therefore p = 2$$

Example 2: In the given parallelogram PQRS, find the lengths of the diagonals PR and QS.



Solution:

We know that the diagonals of a parallelogram bisect each other.

$$\therefore PO = OR$$

$$\Rightarrow 2x - 10 = x$$

$$\Rightarrow 2x - x = 10$$

$$\Rightarrow \therefore x = 10$$

Similarly, QO = OS

$$\Rightarrow y - 2 = 2y - 11$$

$$\Rightarrow 2y - y = -2 + 11$$

$$\Rightarrow \therefore y = 9$$

Now, PR = PO + OR

$$= 2x - 10 + x$$

$$= 3x - 10$$

$$= 3 \times 10 - 10$$

$$= 30 - 10$$

$$= 20$$

Similarly, $QS = QO + OS$

$$= y - 2 + 2y - 11$$

$$= 3y - 13$$

$$= 3 \times 9 - 13$$

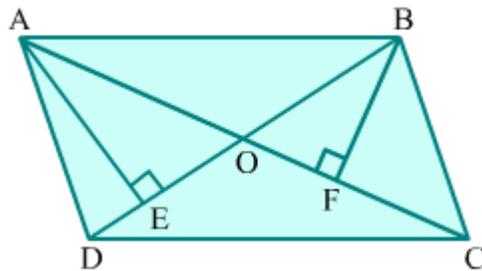
$$= 27 - 13$$

$$= 14$$

Thus, the lengths of the diagonals PR and QS are 20 units and 14 units respectively.

Medium

Example 1: ABCD is a parallelogram with diagonals AC and BD of lengths 10 cm and 8 cm respectively. If the perpendiculars on DO and OC are 5 cm each, then find the sum of the areas of $\triangle AOD$ and $\triangle BOC$.



Solution:

We know that the diagonals of a parallelogram bisect each other.

Therefore, in parallelogram ABCD, we have:

$$AO = OC = \frac{AC}{2} \quad \text{and} \quad BO = OD = \frac{BD}{2}$$

$$\Rightarrow AO = OC = \frac{10}{2} \text{ cm} = 5 \text{ cm} \quad \text{and} \quad BO = OD = \frac{8}{2} \text{ cm} = 4 \text{ cm}$$

$$\text{Now, area of } \triangle AOD = \frac{1}{2} \times \text{Base} \times \text{Height}$$

$$= \frac{1}{2} \times OD \times AE$$

$$= \frac{1}{2} \times 4 \times 5 \text{ cm}^2$$

$$= 10 \text{ cm}^2$$

Similarly, area of $\Delta BOC = \frac{1}{2} \times OC \times BF$

$$= \frac{1}{2} \times 5 \times 5 \text{ cm}^2$$

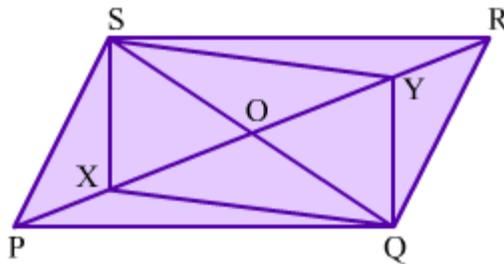
$$= 12.5 \text{ cm}^2$$

Therefore, sum of the areas of ΔAOD and $\Delta BOC = (10 + 12.5) \text{ cm}^2 = 22.5 \text{ cm}^2$

Hard

Example 1: In parallelogram PQRS, X and Y are points on PR such that $PX = YR$. Prove that:

1. **XQYS is a parallelogram**
2. **$\Delta SXP \cong \Delta QYR$**



Solution:

1. We know that the diagonals of a parallelogram bisect each other.

$$\therefore OS = OQ \dots (1)$$

$$\text{And } OP = OR \dots (2)$$

$$\text{Also, } PX = YR \dots (3) \text{ [Given]}$$

On subtracting equation 3 from equation 2, we obtain:

$$OP - PX = OR - YR$$

$$\Rightarrow OX = OY \dots (4)$$

In quadrilateral XQYS, XY and QS are the diagonals.

We know from equations 1 and 4 that the diagonals bisect each other.

Thus, XQYS is a parallelogram.

2. In $\triangle SXP$ and $\triangle QYR$, we have:

$$PS = QR \text{ (Opposite sides of parallelogram PQRS)}$$

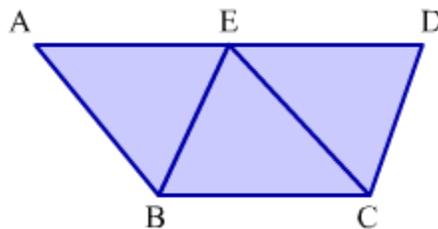
$$SX = QY \text{ (Opposite sides of parallelogram XQYS)}$$

$$PX = YR \text{ (Given)}$$

$$\therefore \triangle SXP \cong \triangle QYR \text{ (By the SSS congruence rule)}$$

A Quadrilateral is a Parallelogram if a Pair of Opposite Sides is Equal and Parallel
Necessary Condition for a Quadrilateral to Be a Parallelogram

Consider the given figure.



Here, ABCE is a parallelogram and AE is extended to D such that $AE = ED$.

Using an important property of parallelograms, we can prove that BCDE is also a parallelogram.

The property used for proving the above can be stated as follows:

A quadrilateral is a parallelogram if it has one pair of parallel and equal (or congruent) opposite sides.

Since ABCE is a parallelogram, $AE \parallel BC$ and $AE = BC$.

Also, $AE = ED \Rightarrow ED = BC$

And $AE \parallel BC \Rightarrow AD \parallel BC \Rightarrow ED \parallel BC$

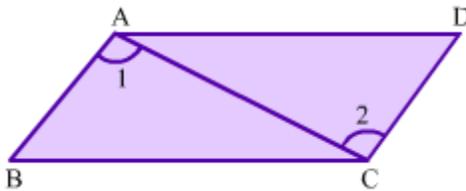
Hence, by the above-stated property, BCDE is a parallelogram.

In this lesson, we will understand and prove this property of parallelograms. We will also solve some examples based on the same.

Solved Examples

Easy

Example 1: In the given figure, $\angle 1 = \angle 2$ and $AB = DC$. Is quadrilateral ABCD a parallelogram?



Solution:

In quadrilateral ABCD, $\angle 1 = \angle 2$ (Given)

These angles are alternate angles made by the transversal AC between lines AB and DC.

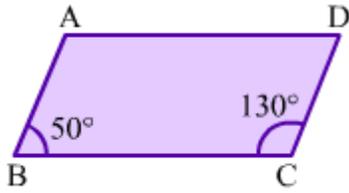
We know that if equal alternate angles are made by a transversal between two lines, then the lines intersected by the transversal are parallel.

$\therefore AB \parallel DC$

Also, $AB = DC$ (Given)

We know that a quadrilateral is a parallelogram if it has a pair of parallel and equal opposite sides. Thus, ABCD is a parallelogram.

Example 2: In the given figure, $AB = DC$. Prove that $AD \parallel BC$.



Solution:

In quadrilateral ABCD, $\angle B + \angle C = 50^\circ + 130^\circ = 180^\circ$

We know that if the interior angles on the same side of a transversal are supplementary, then the lines intersected by the transversal are parallel.

$\Rightarrow AB \parallel DC$

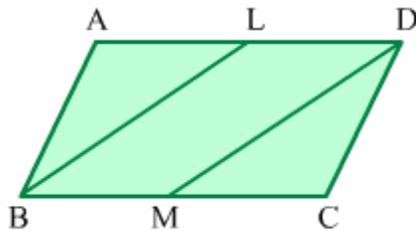
Also, $AB = DC$ (Given)

We know that a quadrilateral is a parallelogram if it has a pair of parallel and equal opposite sides. Thus, ABCD is a parallelogram.

$\Rightarrow AD \parallel BC$ (\because Opposite sides of a parallelogram are parallel)

Medium

Example 1: ABCD is a parallelogram in which L and M are the midpoints of AD and BC respectively. Prove that BMDL is a parallelogram.



Solution:

It is given that L and M are the midpoints of AD and BC respectively.

$$\therefore BM = \frac{1}{2} BC \text{ and } LD = \frac{1}{2} AD$$

ABCD is a parallelogram. (Given)

So, $BC = AD$ (\because Opposite sides of a parallelogram are equal)

$$\therefore \frac{1}{2}BC = \frac{1}{2}AD$$

$$\Rightarrow BM = LD \dots (1)$$

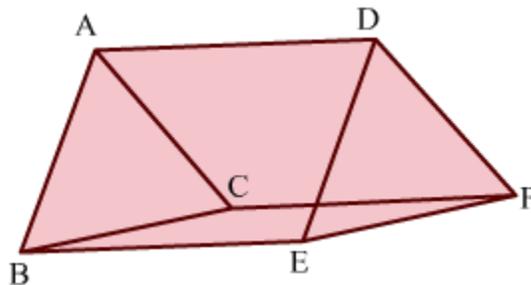
Also, $BC \parallel AD$ (\because Opposite sides of a parallelogram are parallel)

$$\Rightarrow BM \parallel LD \dots (2)$$

We know that a quadrilateral is a parallelogram if it has a pair of parallel and equal opposite sides. From 1 and 2, we conclude that $BMDL$ is a parallelogram.

Hard

Example 1: Sides AB and BC of ΔABC are parallel and equal to the corresponding sides DE and EF of ΔDEF . Prove that $ACFD$ is a parallelogram.



Solution:

Consider quadrilateral $ABED$.

We have $AB = DE$ and $AB \parallel DE$ (Given)

Hence, $ABED$ is a parallelogram. (\because There is one pair of equal and parallel opposite sides)

$$\Rightarrow AD = BE \text{ and } AD \parallel BE \dots (1)$$

Now, consider quadrilateral $BCFE$.

We have $BC = EF$ and $BC \parallel EF$ (Given)

Hence, $BCFE$ is a parallelogram. (\because There is one pair of equal and parallel opposite sides) \Rightarrow
 $CF = BE$ and $CF \parallel BE \dots (2)$

From 1 and 2, we have:

$$AD = CF \text{ and } AD \parallel CF$$

Hence, ACFD is a parallelogram. (\because There is one pair of equal and parallel opposite sides)

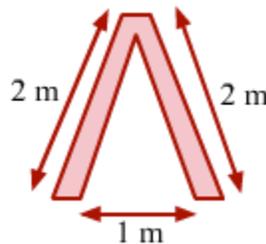
Mid Point Theorem and Its Converse

Midpoint Theorem

The given image is that of a food outlet named Adam's Diner. The logo of the outlet, as you can see on top of the building, is the letter 'A'.



The carpenter who made the logo fixed the two legs of the logo, each of length 2 m, such that the distance between the two legs at the base was 1 m.



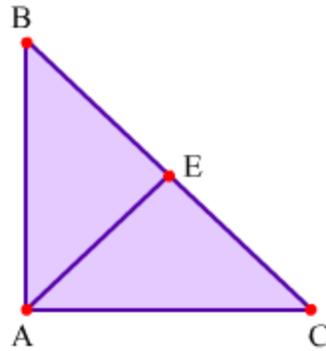
The carpenter then put the horizontal bar of the logo at the **mid-point**

of the two legs so that it fitted exactly in the middle of the legs. The midpoint theorem came to his aid while doing so. In this lesson, we will study this theorem, its proof and its converse, and solve some related examples.

Whiz Kid

Right triangle median theorem

The length of the median on the hypotenuse of a right triangle is half that of the hypotenuse.



Here, $AE = \frac{1}{2} \times BC$

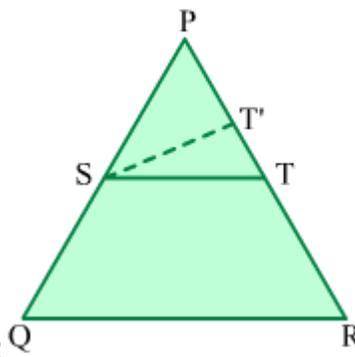
Activity

Perform the following activity to verify the midpoint theorem.

- Draw a large scalene triangle on a sheet of paper.
- Name its vertices A, B and C. Find the midpoints of sides AB and AC and name them D and E respectively. Draw a line joining the two midpoints.
- Cut out $\triangle ABC$ and then cut it along line segment DE.
- Draw a quadrilateral BDEC. Place $\triangle ADE$ on it such that vertices E and D of the triangle are respectively on vertex C and line segment BC of the quadrilateral. Mark the point where vertex D touches line segment BC.
- Shift $\triangle ADE$ such that its vertices D and E are respectively on vertex B and line segment BC of the quadrilateral. Mark the point where vertex E touches line segment BC.
- What do you notice about the lengths of DE and BC? Do the marked points coincide?

Proof of the Converse of the Midpoint Theorem

Statement: The line drawn through the midpoint of one side of a triangle and parallel to another side bisects the third side.



Given: $\triangle PQR$ in which S is the midpoint of PQ and $ST \parallel QR$

To prove: T is the midpoint of PR

Construction: Take a point T' on PR and join S and T' .

Proof: Let us say that T is not the midpoint of PR. Let T' be the midpoint of PR.

In ΔPQR , S is the midpoint of PQ and T' is the midpoint of PR.

\therefore By the midpoint theorem, $ST' \parallel QR$... (1)

It is given that $ST \parallel QR$ (2)

From 1 and 2, we conclude that $ST \parallel ST'$, which is not possible.

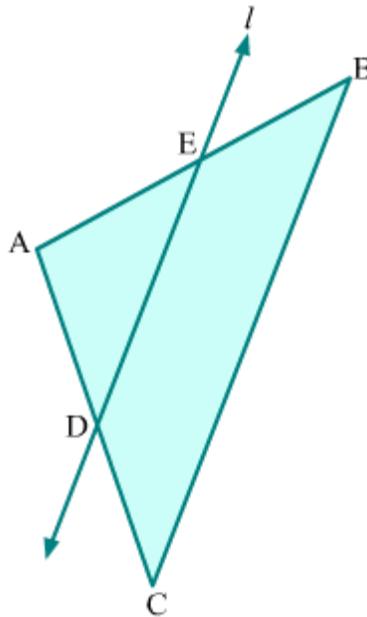
So, our assumption was wrong.

Hence, T is the midpoint of PR.

Solved Examples

Easy

Example 1: Line l cuts the isosceles ΔABC , with $AB = AC = 10$ cm and $BC = 15$ cm, in such a way that ΔADE is also isosceles with $AD = AE = 5$ cm. Find the length of DE.



Solution:

It is given that $\triangle ABC$ is isosceles with $AB = AC = 10$ cm, $BC = 15$ cm and $AD = AE = 5$ cm.

$$AC = AD + DC$$

$$\Rightarrow DC = AC - AD = (10 - 5) \text{ cm} = 5 \text{ cm}$$

Since $AD = DC$, D is the midpoint of AC .

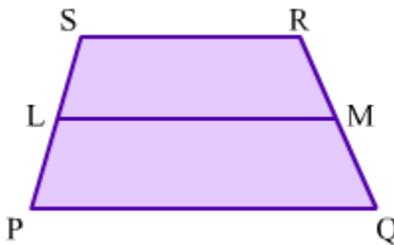
Similarly, E is the midpoint of AB .

Using the midpoint theorem, we get:

$$DE = \frac{BC}{2} = \frac{15}{2} \text{ cm} = 7.5 \text{ cm}$$

Medium

Example 1: In the given trapezium $PQRS$, $PQ \parallel SR$ and L is the midpoint of line segment PS . If $LM \parallel PQ$, then prove that $RM = QM$.

**Solution:**

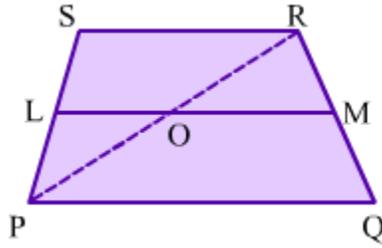
It is given that:

$$LM \parallel PQ$$

$$PQ \parallel SR$$

$$\Rightarrow LM \parallel SR (\because \text{Lines parallel to the same line are also parallel})$$

Construction: Join P to R such that PR intersects LM at point O .



In $\triangle PSR$, L is the midpoint of PS and $LO \parallel RS$.

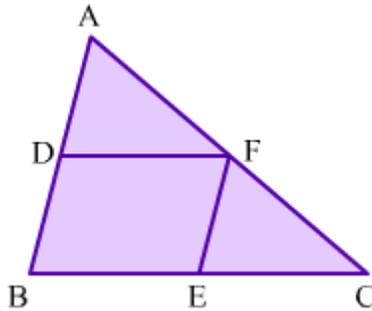
So, by the converse of the midpoint theorem, we get O as the midpoint of PR.

Now, in $\triangle PQR$, O is the midpoint of PR and $OM \parallel PQ$.

Again, by the converse of the midpoint theorem, we get M as the midpoint of QR.

$$\Rightarrow RM = QM$$

Example 2: In the shown $\triangle ABC$, D, E and F are the midpoints of sides AB, BC and AC respectively. Prove that BEFD is a parallelogram.



Solution:

In $\triangle ABC$, D and F are the midpoints of AB and AC respectively.

So, by the midpoint theorem, we obtain $DF \parallel BC$

$$\Rightarrow DF \parallel BE \dots (1)$$

Again, in $\triangle ABC$, E and F are the midpoints of BC and AC respectively.

So, by the midpoint theorem, we obtain $EF \parallel BA$

$$\Rightarrow EF \parallel BD \dots (2)$$

Now in quadrilateral BEFD, we have:

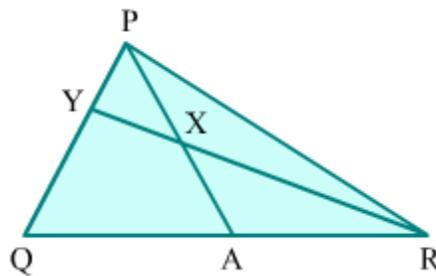
$DF \parallel BE$ and $EF \parallel BD$ (From 1 and 2)

\Rightarrow BEFD is a parallelogram. (\because There are two pairs of parallel opposite sides)

Hard

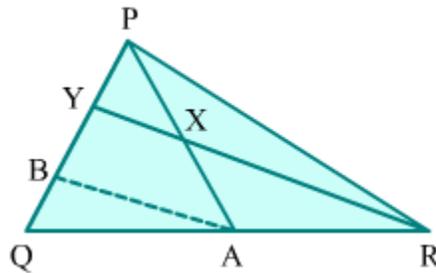
Example 1: In the given ΔPQR , PA is the median to side QR and X is the midpoint of

PA. When produced, RX meets PQ at Y. Prove that $PY = \frac{1}{3}PQ$.



Solution:

Construction: Draw a line AB such that it is parallel to RY.



In ΔPAB , X is the midpoint of AP and $XY \parallel AB$.

So, by the converse of the midpoint theorem, we obtain Y as the midpoint of PB.

$\Rightarrow PY = YB \dots (1)$

In ΔQYR , A is the midpoint of QR. (\because AP is the median to side QR)

Also, $RY \parallel AB$

So, by the converse of the midpoint theorem, we obtain B as the midpoint of YQ.

$$\Rightarrow YB = BQ \dots (2)$$

From equations 1 and 2, we obtain:

$$PY = YB = BQ \dots (3)$$

From the figure, we have:

$$PQ = PY + YB + BQ$$

$$\Rightarrow PQ = PY + PY + PY$$

$$\Rightarrow PQ = 3PY$$

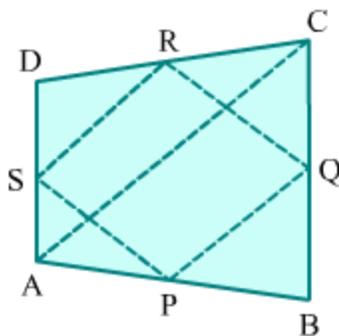
$$\Rightarrow \therefore PY = \frac{1}{3}PQ$$

Example 2: Prove that the quadrilateral formed by joining the midpoints of all sides of a quadrilateral is a parallelogram.

Solution:

Let ABCD be a quadrilateral and P, Q, R and S the respective midpoints of sides AB, BC, CD and DA.

Construction: Draw diagonal AC of quadrilateral ABCD.



To prove that PQRS is a parallelogram, we have to prove that it has one pair of parallel and equal opposite sides.

In $\triangle ACD$, R and S are the midpoints of sides CD and DA respectively.

So, by the midpoint theorem, we obtain $SR \parallel AC$ and $SR = \frac{1}{2}AC \dots (1)$

Now, in $\triangle ABC$, P and Q are the midpoints of sides AB and BC respectively.

So, by midpoint theorem, we obtain $PQ \parallel AC$ and $PQ = \frac{1}{2} AC$... (2)

From 1 and 2, we obtain:

$PQ \parallel RS$ (\because Lines parallel to the same line are also parallel)

$PQ = RS$

In quadrilateral PQRS, we have one pair of parallel and equal opposite sides, i.e., $PQ \parallel RS$ and $PQ = RS$. Thus, PQRS is a parallelogram.