Real Numbers

Euclids Division Lemma

What is a dividend? Let us understand it with the help of a simple example.

Can you divide 14 by 6?

$$\begin{array}{r} 2 \\ 6\overline{)14} \\ -12 \\ \hline 2 \end{array}$$

After division, we get 2 as the quotient and 2 as the remainder.

Thus, we can also write 14 as 6 × 2 + 2.

A dividend can thus be written as:

Dividend = Divisor × Quotient + Remainder

Can you think of any other number which, when multiplied with 6, gives 14 as the dividend and 2 as the remainder?

Let us try it out with some other sets of dividends and divisors.

(1) Divide 100 by 20: 100 = 20 × 5 + 0

(2) Divide 117 by 15: 117 = 15 × 7 + 12

(3) Divide 67 by 17: 67 = 17 × 3 + 16

Thus, if we have a dividend and a divisor, then there will be a unique pair of a quotient and a remainder that will fit into the above equation.

This brings us to **Euclid's division lemma.**

If *a* and *b* are positive integers, then there exist two unique integers, *q* and *r*,

such that $a = bq + r$

This lemma is very useful for finding the H.C.F. of large numbers where breaking them into factors is difficult. This method is known as **Euclid's Division Algorithm**.

Let us look at some more examples.

Example 1: Find the H.C.F. of 4032 and 262 using Euclid's division algorithm.

Solution:

Step 1:

First, apply Euclid's division lemma on 4032 and 262.

$4032 = 262 \times 15 + 102$

Step 2:

As the remainder is non-zero, we apply Euclid's division lemma on 262 and 102.

$262 = 102 \times 2 + 58$

Step 3:

Apply Euclid's division lemma on 102 and 58.

$102 = 58 \times 1 + 44$

Step 4:

Apply Euclid's division lemma on 58 and 44.

$58 = 44 \times 1 + 14$

Step 5:

Apply Euclid's division lemma on 44 and 14.

$44 = 14 \times 3 + 2$

Step 6:

Apply Euclid's division lemma on 14 and 2.

$14 = 2 \times 7 + 0$

In the problem given above, to obtain 0 as the remainder, the divisor has to be taken as 2. Hence, 2 is the H.C.F. of 4032 and 262.

Note that Euclid's division algorithm can be applied to polynomials also.

Example 2: A rectangular garden of dimensions 190 m × 60 m is to be divided in square blocks to plant different flowers in each block. Into how many blocks can this garden be divided so that no land is wasted?

Solution:

If we do not want to waste any land, we need to find the largest number that completely divides both 190 and 60 and gives the remainder 0, i.e., the H.C.F. of (190, 60).

To find the H.C.F., let us apply Euclid's algorithm.

$190 = 60 \times 3 + 10$

$60 = 10 \times 6 + 0$

Therefore, the H.C.F. of 190 and 60 is 10.

Therefore, there will be $\frac{190}{10} = 19$ square blocks along the length of the garden and $\frac{60}{10} = 6$ blocks along its breadth.

Hence, the total number of blocks in the garden will be $19 \times 6 = 114$.

Example 3: Find the H.C.F. of 336 and 90 using Euclid's division algorithm.

Solution:

As 336 > 90, we apply the division lemma to 336 and 90.

$336 = 90 \times 3 + 66$

Applying Euclid's division lemma to 90 and 66:

$90 = 66 \times 1 + 24$

Applying Euclid's division lemma to 66 and 24:

66 = 24 × 2 + 18

Applying Euclid's division lemma to 24 and 18:

24 = 18 × 1 + 6

Applying Euclid's division lemma to 18 and 6:

18 = 6 × 3 + 0

As the remainder is zero, we need not apply Euclid's division lemma anymore. The divisor (6) is the required H.C.F.

Example 4: Find the H.C.F. of 45, 81, and 117 using Euclid's division algorithm.

Solution:

Let us begin by choosing any two out of the three given numbers, say 45 and 81.

As 81 > 45, we apply Euclid's division lemma to 81 and 45.

81 = 45 × 1 + 36

Applying Euclid's division lemma to 45 and 36:

45 = 36 × 1 + 9

Applying Euclid's division lemma to 36 and 9:

36 = 9 × 4 + 0

As the remainder is zero, the H.C.F. of 45 and 81 is 9.

Now, we again need to apply Euclid's division algorithm on the H.C.F. of the two numbers and the remaining number.

Since the H.C.F. of 45 and 81 is 9 and the third number is 117, we apply Euclid's division lemma to 117 and 9.

117 = 9 × 13 + 0

As the remainder is zero, the H.C.F. of 9 and 117 is 9.

Here, the second H.C.F. (the H.C.F. of the H.C.F. of the first two numbers and the third number) is the H.C.F. of the three numbers.

Thus, we can say that the H.C.F. of the three numbers is 9.

Example 5: In an inter-school essay writing competition, the numbers of participants from schools A, B, and C are 20, 16, and 28 respectively. If the participants in each room are from the same school, then find the minimum number of rooms required such that each room has the same number of participants.

Solution:

If we need to find the minimum number of rooms, then we need to keep the maximum number of participants in each room i.e., we need to find the largest number that completely divides 20, 16, and 28.

Thus, we start by choosing any two out of the given three numbers, say 20 and 16.

Applying Euclid's division lemma to 20 and 16:

$20 = 16 \times 1 + 4$

Applying Euclid's division lemma to 16 and 4:

$16 = 4 \times 4 + 0$

Hence, 4 is the H.C.F. of 20 and 16.

Applying Euclid's division algorithm to 28 and 4:

$28 = 4 \times 7 + 0$

Hence, 4 is the H.C.F. of 28 and 4.

∴H.C.F. of 20, 16, 28 = 4

Therefore, each room would have 4 participants.

The number of rooms in which the participants from schools A, B, and C can be accommodated is $\frac{20}{4} = 5$, $\frac{16}{4} = 4$, and $\frac{28}{4} = 7$ respectively.

Therefore, a total of 5 + 4 + 7 =16 rooms are required.

The Use Of Euclids Division Lemma To Prove Mathematical Relationships

Whenever we divide 32 by 5, we will get 6 as the quotient and 2 as the remainder. Therefore, we can also write 32 as $6 \times 5 + 2$.

We can thus write this rule about the dividend as

Dividend = Divisor $\times$ Quotient + Remainder

This expression is unique, i.e., whenever we divide an integer (dividend) by another integer (divisor), we always get a fixed quotient and a fixed remainder.

The above statement can be proven by taking an example.

Whenever we divide 54 by 11, we will get 4 as the quotient and 10 as the remainder. We will never get a quotient other than 4 and a remainder other than 10 when we divide 54 by 11.

This brings us to **Euclid's division lemma**.

If a and b are positive integers, then there exist integers q and r such that

$$a = bq + r, \text{ where } 0 \leq r < b$$

This lemma has several applications, one of which is to prove mathematical relationships among numbers.

Let us discuss this concept with the help of a few examples.

Example 1: Prove that every positive integer is of the form $3p$, $3p + 1$, or $3p + 2$, where p is any integer.

Solution:

Let a be any positive integer and let $b = 3$.

Applying Euclid's algorithm to a and 3:

$a = 3p + r$; for some integer p and $0 \le r < 3$

Therefore, a can be $3p$, $3p + 1$, or $3p + 2$.

As a is a positive integer, we can say that any positive integer is of the form $3p$, $3p + 1$, or $3p + 2$.

Example 2: Prove that every positive even integer is of the form $2m$ and every positive odd integer is of the form $2m + 1$, where m is any integer.

Solution:

Let a be any positive integer and let $b = 2$.

According to Euclid's division lemma, there exist two unique integers m and r such that

$a = bm + r = 2m + r$, where $0 \le r < 2$.

Thus, $r = 0$ or 1

If $r = 0$, i.e., if $a = 2m$, then the expression is divisible by 2. Thus, it is an even number.

If $r = 1$, i.e., if $a = 2m + 1$, then the expression is not divisible by 2. Thus, it is an odd number.

Thus, every positive even integer is of the form $2m$ and every positive odd integer is of the form $2m + 1$.

Example 3: Prove that the expression $y(y + 1)$ always represents an even number, where y is any positive integer.

Solution:

Let y be an integer. Thus, it may be either odd or even.

When y is an odd number, i.e., when y is of the form $2p + 1$, where p is an integer:

$y(y + 1) = (2p + 1)(2p + 1 + 1) = (2p + 1)(2p + 2) = 2(2p + 1)(p + 1)$

The above expression is a multiple of 2. Thus, the expression $y(y + 1)$ represents an even number.

When y is an even number, i.e., when y is of the form $2q$, where q is an integer:

$y(y + 1) = (2q)(2q + 1) = 2q(2q + 1)$

The above expression is a multiple of 2. Thus, the expression $y(y + 1)$ represents an even number.

Thus, for a positive integer y, the expression $y(y + 1)$ always represents an even number.

Example 4:

Show that

(a) The sum, the difference, and the product of two even numbers is always

even.

(b) The sum and the difference of two odd numbers is always even, whereas

their product is always odd.

Solution:

(a) Let there be two even numbers, x and y, such that $x = 2m$ and $y = 2n$, where m and n are two positive integers.

Now, $x + y = 2m + 2n = 2(m + n) = 2p$, where $p = m + n$ is an integer.

Thus, $x + y$ is always even.

Similarly, $x - y = 2m - 2n = 2(m - n) = 2q$, where $q = m - n$ is an integer.

Thus, $x - y$ is always even.

Likewise, $xy = 2m \times 2n = 4mn = 2(2mn) = 2t$, where $t = 2mn$ is an integer.

Thus, xy is always even.

Hence, the sum, the difference, and the product of two even numbers is always even.

(b) Let there be two odd numbers x and y such that $x = 2m + 1$ and $y = 2n + 1$, where m and n are two positive integers.

Now, $x + y = (2m + 1) + (2n + 1) = 2(m + n + 1) = 2p$, where $p = m + n + 1$ is an integer.

Thus, $x + y$ is even.

Similarly, $x - y = (2m + 1) - (2n + 1) = 2(m - n) = 2q$, where $q = m - n$ is an integer.

Thus, $x - y$ is even.

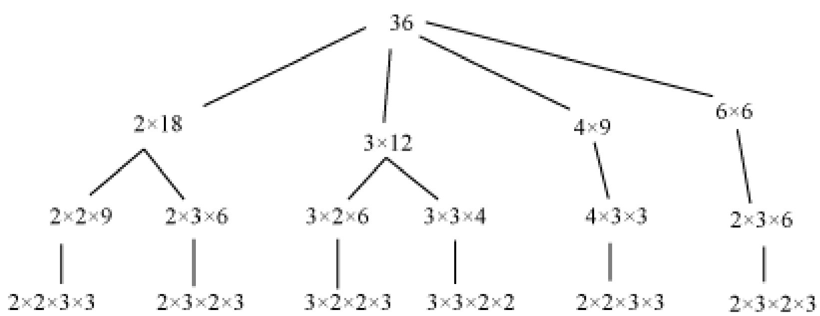
Likewise, $xy = (2m + 1)(2n + 1) = 2(2mn + m + n) + 1 = 2t + 1$, where $t = 2mn + m + n$ is an integer.

Thus, xy is always odd.

Hence, the sum and the difference of two odd numbers is always even, whereas their product is always odd.

Prime Factorisation of Numbers Using Fundamental Theorem of Arithmetic

We know that all composite numbers can be represented as the product of two or more prime numbers. Let us understand this concept by taking the example of 36 and factorising it in different ways.



We can see that whichever way we factorise the number 36, it will be broken down as the product of the same prime numbers, which is unique. The only difference is that the ordering of the prime numbers will be different for different ways of factorising the number. In fact, this is true for all numbers. We can check this by taking the example of a larger number, say 21560, which can be uniquely broken down into its prime factors as

$2^3 \times 5 \times 7^2 \times 11$

Hence, we can say that any composite number can be written in the form of the product of prime numbers, which is unique, except the order in which they occur. By this, we mean that $2 \times 3 \times 7 \times 11$ is the same as $7 \times 11 \times 2 \times 3$.

This is the **fundamental theorem of arithmetic**. It can be formally stated as:

Every composite number can be factorised as the product of certain prime numbers and this factorisation is unique for that composite number although the order in which the prime numbers occur may be changed.

Thus, this theorem can be used to write the prime factorisation of any number. Let us try to build on this concept with the help of some examples.

Example 1:

Write the prime factorization of 31250. What are its prime factors?

Solution:

$31250 = 2 \times 15625$

$= 2 \times 5 \times 3125$

$= 2 \times 5 \times 5 \times 625$

$= 2 \times 5 \times 5 \times 5 \times 125$

$= 2 \times 5 \times 5 \times 5 \times 5 \times 25$

$= 2 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$

$= 2 \times 5^6$

Hence, 2×5^6 is the prime factorisation of 31250. Its prime factors are 2 and 5.

Example 2: If it is given that $13125 = 2^a \times 3^b \times 5^c \times 7^d$, then find the value of $a + 2b + 7c + 11d$.

Solution:

$2^a \times 3^b \times 5^c \times 7^d = 13125$

$= 3 \times 4375$

$= 3 \times 5 \times 875$

$= 3 \times 5 \times 5 \times 175$

$= 3 \times 5 \times 5 \times 5 \times 35$

$= 3 \times 5 \times 5 \times 5 \times 5 \times 7$

$= 3^1 \times 5^4 \times 7^1$

$\therefore 2^a \times 3^b \times 5^c \times 7^d = 2^0 \times 3^1 \times 5^4 \times 7^1$

Comparing exponents of the bases (integers): $a = 0$, $b = 1$, $c = 4$, and $d = 1$

Hence, $a + 2b + 7c + 11d = 0 + 2 \times 1 + 7 \times 4 + 11 \times 1$

$= 0 + 2 + 28 + 11$

$= 41$

Example 3: Show that the expressions given below are composite numbers.

(a) $3 \times 5 \times 7 \times 23 + 2 \times 7 \times 11 \times 13$

(b) $29 \times 35 + 14$

(c) $3^4 + 6^3$

Solution:

(a) $3 \times 5 \times 7 \times 23 + 2 \times 7 \times 11 \times 13 = 7(3 \times 5 \times 23 + 2 \times 11 \times 13)$

= $7(345 + 286)$

$= 7 \times 631$

Since both 7 and 631 are prime numbers, we have expressed the given expression as the product of two prime numbers. We know that according to the fundamental theorem of arithmetic, every composite number can be uniquely written as the product of its prime factors. Thus, the given expression represents a composite number.

(b) $29 \times 35 + 14 = 29 \times 5 \times 7 + 2 \times 7$

$= 7(29 \times 5 + 2)$

$= 7 \times 147$

$= 7 \times 3 \times 7 \times 7$

$= 3 \times 7^3$

Since both 3 and 7 are prime numbers, we have expressed the given expression as the product of its prime factors. We know that according to the fundamental theorem of arithmetic, every composite number can be uniquely written as the product of its prime factors. Thus, the given expression represents a composite number.

(c) $3^4 + 6^3 = 3^4 + (2 \times 3)^3$

= $3^4 + 2^3 \times 3^3$

$= 3^3(3 + 2^3)$

$= 3^3 \times 11$

Since both 3 and 11 are prime numbers, we have expressed the given expression as the product of its prime factors. It is known that according to the fundamental theorem of arithmetic, every composite number can be uniquely written as the product of its prime factors. Thus, the given expression represents a composite number.

Application Of The Fundamental Theorem Of Arithmetic To Find The HCF And LCM Of Numbers

All composite numbers can be written as the product of two or more prime numbers. For example, 20 can be written as $2^2 \times 5$; 54 can be written as 2×3^3, and so on.

Note that if we do not consider the way in which the prime factors are written, then we can prime factorise every number in only one way. This applies to other numbers as well.

This leads to the **fundamental theorem of arithmetic**, which states that:

Every composite number can be factorised as the product of certain prime numbers and this factorisation is unique, although the order in which the prime factors occur may be changed.

Even though we did not notice it before, whenever we prime factorise a number, we use the fundamental theorem of arithmetic to do so.

For example, the prime factorisation of 980 is represented as

$980 = 2 \times 490$

$= 2 \times 2 \times 245$

$= 2 \times 2 \times 5 \times 49$

$= 2 \times 2 \times 5 \times 7 \times 7$

$= 2^2 \times 5^1 \times 7^2$

Hence, $2^2 \times 5^1 \times 7^2$ is the prime factorisation of 980; and 2, 5, and 7 are its prime factors.

By applying the fundamental theorem of arithmetic to the prime factorized numbers, we can also find their HCF and LCM.

This is known as the **prime factorisation method**, which states that:

For any two positive integers *a* and *b*:

HCF (a, b) = Product of the smallest power of each common prime factor in the prime factorisation of numbers

LCM (a, b) = Product of the greatest power of each prime factor in the prime factorisation of numbers

Let us understand this method with the help of some examples.

Example 1: Find the LCM and the HCF of 432 and 676 using the prime factorization method.

Solution:

We can write these numbers as

$432 = 2^4 \times 3^3$

$676 = 2^2 \times 13^2$

To calculate the HCF

We observe that the only common prime factor is 2 and the smallest power of this prime factor is also 2.

Thus, HCF (432, 676) $= 2^2 = 4$

To calculate the LCM

We observe that the prime factors of 432 and 676 are 2, 3, and 13. The greatest powers of these factors are 4, 3, and 2 respectively.

LCM is the product of the greatest power of each prime factor.

Thus, LCM (432, 676) = $2^4 \times 3^3 \times 13^2 = 73008$

Example 2: Find the HCF and the LCM of 28, 42, and 64 using the prime factorization method.

Solution:

We can write these numbers as

$28 = 2^2 \times 7^1$

$42 = 2 \times 3^1 \times 7^1$

$64 = 2^6$

HCF is the product of the smallest power of each common prime factor.

Here, the only common prime factor is 2 and its power is 1.

Thus, HCF (28, 42, 64) = $2^1 = 2$

LCM is the product of the greatest power of each prime factor.

Thus, LCM (28, 42, 64) = $2^6 \times 3^1 \times 7^1 = 1344$

Example 3: Find the HCF and the LCM of 1080 and 900 using the prime factorization and show that HCF × LCM = Product of two numbers.

Solution:

$1080 = 2^3 \times 3^3 \times 5$

$900 = 2^2 \times 3^2 \times 5^2$

Hence, HCF (1080, 900) = $2^2 \times 3^2 \times 5 = 180$

LCM (1080, 900) = $2^3 \times 3^3 \times 5^2 = 5400$

HCF × LCM = 180 × 5400 = 972000

Product of numbers = 1080 × 900 = 972000

Hence, HCF × LCM = Product of two numbers

Example 4: The HCF of 273 and another number is 7, while their LCM is 3003. Find the other number.

Solution:

Let the first number (a) be 273 and the second number be b.

It is given that HCF (a, b) = 7 and LCM (a, b) = 3003.

We know that HCF × LCM = Product of two numbers.

⇒ HCF (a, b) × LCM (a, b) = $a \times b$

⇒ 7 × 3003 = 273 × b

$$\Rightarrow b = \frac{7 \times 3003}{273}$$

⇒ b = 77

Hence, the other number is 77.

Example 5: Anurag takes 6 minutes to complete one round of jogging around the circular track of a park, while Twinkle takes 8 minutes to do the same. If both of them start jogging at the same time from the same point, then how much time will they take before they meet at the point from which they started?

Solution:

Since Anurag and Twinkle take 6 minutes and 8 minutes respectively to complete one round of the circular track, the time after which they will meet at the starting point will be the lowest multiple of 6 and 8, i.e., their LCM.

6 = 2 × 3

8 = 2^3

∴ LCM (6, 8) = $2^3 \times 3$ = 24

Thus, they will meet at the starting point after 24 minutes.

Example 6: There are 120 students in a class. When the students were arranged according to their roll numbers, it was observed that every second student got distinction in Mathematics, every third student got distinction in Science, and every fifth student got distinction in English. How many students got distinction in all three subjects?

Solution:

As every second, third, and fifth student got distinction in Math, Science, and English respectively, the roll numbers of the students who got distinction in all three subjects will be equal to the multiples of the LCM of 2, 3, and 5.

LCM (2, 3, 5) = 2 × 3 × 5 = 30

Thus, every 30th student got distinction in all three subjects.

Thus, a total of $\frac{120}{30} = 4$ students got distinction in all three subjects.

Properties Of Prime Numbers

Consider the number 8^n, where n is a natural number.

Is there any value of *n* for which 8^n ends with zero?

It is difficult to answer this question directly. However, we can answer this question by making use of the fundamental theorem of arithmetic. It states that

> **"Every composite number can be factorized as a product of primes and this factorization is unique, apart from the order in which the prime factors occur".**

This means that if we are given a composite number, then that number can be written as a product of prime numbers in only one way (except for the order of prime numbers).

For example: the composite number 255 can be written as the product of primes as follows.

255 = 3 × 5 × 17

Also, 255 can be written as 3 × 17 × 5 or 5 × 3 × 17 or 5 × 17 × 3 or 17 × 3 × 5 or 17 × 5 × 3.

Thus, we can see that 255 can be expressed as a product of unique prime numbers 3, 5, and 17 but the order of representation may differ.

Now, by making use of the above theorem, we can answer the question which we were discussing in the beginning. Let us see how.

Suppose the number 8^n ends with zero for some value of n.

Since the number ends with zero, it should be divisible by 10.

Now, $10 = 2 \times 5$

Thus, this number should be divisible by 2 and 5 also.

Therefore, the prime factorization of 8^n should contain both the prime numbers 2 and 5.

We have, $8^n = (2^3)^n = 2^{3n}$

⇒ The only prime in the factorization of 8^n is 2.

Thus, by fundamental theorem of arithmetic, there is no other prime in the factorization of 8^n.

Hence, there is no natural number n for which 8^n ends with the digit zero.

In this way, we can make use of the above theorem.

Let us now look at some more examples to understand this concept better.

Example 1: Prove that the number 9^n, where n is a natural number, cannot end with a zero.

Solution:

Suppose the number 9^n ends with a zero for some value of n.

Since the number ends with zero, it should be divisible by 10.

Now, $10 = 2 \times 5$

Thus, this number should be divisible by 2 and 5 also.

Therefore, the prime factorization of 9^n should contain both the prime numbers 2 and 5.

We have, $9^n = (3^2)^n = 3^{2n}$

⇒ The only prime in the factorization of 9^n is 3.

Thus, by fundamental theorem of arithmetic, there is no other prime in the factorization of 9^n.

Hence, there is no natural number n for which 9^n ends with the digit zero.

Example 2: Check whether the numbers 49^n, where n is a natural number, can end with a zero.

Solution:

Suppose the number 49^n ends with a zero for some value of n.

Since the number ends with zero, it should be divisible by 10.

Now, $10 = 2 \times 5$

Thus, this number should be divisible by 2 and 5 also.

Therefore, the prime factorization of 49^n should contain both the prime numbers 2 and 5.

We have, $49^n = (7^2)^n = 7^{2n}$

⇒ The only prime in the factorization of 49^n is 7.

Thus, by fundamental theorem of arithmetic, there is no other prime in the factorization of 49^n.

Thus, there is no natural number n for which 49^n ends with the digit zero.

Irrational Numbers

We know that a number which cannot be written in the form of $\frac{p}{q}$, where p and q are integers and q ≠ 0, is known as an irrational number.

For example: all numbers of the form $\sqrt{p}$, where p is a prime number such as $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}$ etc., are irrational numbers.

How can we prove that these are irrational numbers?

We can prove this by making use of a theorem which can be stated as follows.

"If *p* divides *a*², then *p* divides *a* (where *p* is a prime number and *a* is a positive integer)".

Proof of theorem: Let $a = p_1 \times p_2 \times \ldots \times p_n$ ...(1)
where $p_1, p_{2,\ldots} p_n$ are the prime factors of a.
Squaring (1) we get
$a^2 = (p_1 \times p_2 \times \ldots\ldots \times p_n)^2$
But p divides a^2
p is a factor of ${p_1}^2 \times {p_2}^2 \times \ldots\ldots \times {p_n}^2$
By the fundamental theorem of arithmetic, the primes in the factorisation of ${p_1}^2 \times {p_2}^2 \times \ldots\ldots \times {p_n}^2$ are unique.
So, p is one out of the primes $p_1, p_{2,\ldots} p_n$.
If $p = p_k$ where k has value from 1 to n, p_k divides $p_1 \times p_2 \times \ldots\ldots p_n$
So, p_k divides a.
Thus, p divides a as $p_k = p$.

So go through the given video to understand the application of the above stated property.

Similarly, we can prove that square roots of other prime numbers like $\sqrt{3}, \sqrt{5}, \sqrt{11}$, etc. are irrational numbers.

Besides these irrational numbers, there are some other irrational numbers like $9\sqrt{2}, 5\sqrt{7}$ etc.

We can also prove why these numbers are irrational. Before this, let us first see what happens to irrational numbers, when we apply certain mathematical operations on them.

Addition or subtraction of two irrational numbers gives a rational or an irrational number.
Addition or subtraction of a rational and an irrational number gives an irrational number.
Multiplication of a non-zero rational number and an irrational number gives an irrational number.
Multiplication of two irrational numbers gives a rational or an irrational number.

We will now prove that $9\sqrt{2}$ **is irrational.**

We know that $\sqrt{2}$ is irrational (as proved before).

Now, the multiplication of a rational and an irrational number gives an irrational number.

Therefore, $9\sqrt{2}$ is an irrational number.

Let us now try to understand the concept further through some more examples.

Example 1: Prove that $5+2\sqrt{7}$ **is irrational.**

Solution:

Let us assume that $5+2\sqrt{7}$ is not irrational, i.e. $5+2\sqrt{7}$ is a rational number.

Then we can write $5+2\sqrt{7}=\frac{a}{b}$, where *a* and *b* are integers and*b* ≠ 0.

Let *a* and *b* have a common factor other than 1.

After dividing by the common factor, we obtain

$5+2\sqrt{7}=\frac{c}{d}$, where *c* and *d* are co-prime numbers.

$$\Rightarrow 5-\frac{c}{d}=-2\sqrt{7}$$

$$\Rightarrow 2\sqrt{7}=\frac{c}{d}-5$$

$$\Rightarrow \sqrt{7}=\frac{c}{2d}-\frac{5}{2}$$

As *c*, *d* and 2 are integers, $\frac{c}{2d}$ and $\frac{5}{2}$ are rational numbers.

Thus, $\frac{c}{2d}-\frac{5}{2}$ is rational.

$\Rightarrow \sqrt{7}$ is rational as the difference of two rational numbers is again a rational number.

This is a contradiction as $\sqrt{7}$ is irrational.

Therefore, our assumption that $5+2\sqrt{7}$ is rational is wrong.

Hence, $5+2\sqrt{7}$ is irrational.

Example 2: Prove that $3-\sqrt{5}$ is irrational.

Solution:

Let us assume $3-\sqrt{5}$ is rational. Then, we can write

$3-\sqrt{5}=\frac{a}{b}$,

where *a* and *b* are co-prime and *b* ≠ 0.

$\Rightarrow \sqrt{5}=3-\frac{a}{b}$

Now, as *a* and *b* are integers, $\frac{a}{b}$ is rational or $3-\frac{a}{b}$ is a rational number.

This means that $\sqrt{5}$ is rational.

This is a contradiction as $\sqrt{5}$ is irrational.

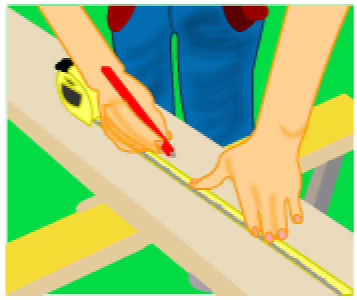
Therefore, our assumption that $3-\sqrt{5}$ is rational is wrong.

Hence, $3-\sqrt{5}$ is an irrational number.

Decimal Expansions of Rational Numbers

The Need for Converting Rational Numbers into Decimals

A carpenter wishes to make a point on the **edge** of a wooden plank at 95 mm from any end. He has a centimeter tape, but how can he use that to mark the required point?



Simple! He should convert 95 mm into its corresponding centimeter value, i.e., 9.5 cm and then measure and mark the required length on the wooden plank.

This is just one of the many situations in life when we face the need to convert numbers into decimals. In this lesson, we will learn to convert rational numbers into decimals, observe the types of decimal numbers, and solve a few examples based on this concept.

Know More

Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if $ad = bc$.

Take, for example, the rational numbers $\frac{2}{4}$ and $\frac{3}{6}$. Let us see if they are equal or not.

Here, $a = 2$, $b = 4$, $c = 3$ and $d = 6$

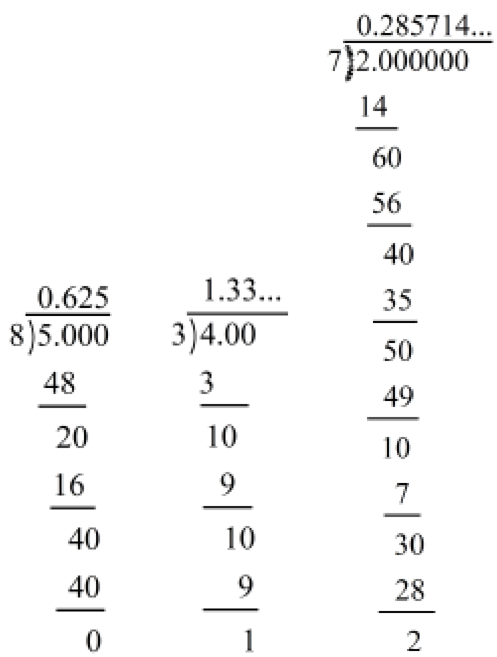
Now, we have:

$ad = 2 \times 6 = 12$

$bc = 4 \times 3 = 12$

Since $ad = bc$, we obtain $\frac{2}{4} = \frac{3}{6}$.

We know that the form $\frac{p}{q}$ represents the division of integer p by the integer q. By solving this division, we can find the decimal equivalent of the rational number $\frac{p}{q}$. Now, let us convert the numbers $\frac{5}{8}$, $\frac{4}{3}$ and $\frac{2}{7}$ into decimals using the long division method.



While the remainder is zero in the division of 5 by 8, it is not so in case of the other two divisions. Thus, we can get two different cases in the decimal expansions of rational numbers.

Observing the Decimal Expansions of Rational Numbers

We can get the following two cases in the decimal expansions of rational numbers.

Case I: When the remainder is zero

In this case, the remainder becomes zero and the quotient or decimal expansion terminates after a finite number of digits after the decimal point. For example, in the decimal expansion of $\frac{5}{8}$, we get the remainder as zero and the quotient as 0.625.

Case II: When the remainder is never zero

In this case, the remainder never becomes zero and the corresponding decimal expansion is non-terminating. For example, in the decimal expansions of $\frac{4}{3}$ and $\frac{2}{7}$, we see that the remainder never becomes zero and their corresponding quotients are **non-terminating decimals**.

When we divide 4 by 3 and 2 by 7, we get 1.3333… and 0.285714285714… as the respective quotients. In these decimal numbers, the digit ‘3’ and the group of digits

'285714' get repeated. Therefore, we can write $\frac{4}{3} = 1.3333... = 1.\overline{3}$ and $\frac{2}{7} = 0.285714285714... = 0.\overline{285714}$. Here, the symbol $\overline{\quad}$ indicates the digit or group of digits that gets repeated.

Solved Examples

Example 1: Write the decimal expansion of $\mathbf{\frac{1237}{25}}$ and find if it is terminating or non-terminating and repeating.

Solution:

Here is the long division method to find the decimal expansion of $\frac{1237}{25}$.

$$\begin{array}{r} 49.48 \\ 25\overline{)1237.00} \\ \underline{100} \\ 237 \\ \underline{225} \\ 120 \\ \underline{100} \\ 200 \\ \underline{200} \\ 0 \end{array}$$

Hence, the decimal expansion of $\frac{1237}{25}$ is 49.48. Since the remainder is obtained as zero, the decimal numberis terminating.

Example 2: Write the decimal expansion of $\mathbf{\frac{2358}{27}}$ and find if it is terminating or non-terminating and repeating.

Solution:

Here is the long division method to find the decimal expansion of $\frac{2358}{27}$.

$$
\begin{array}{r}
87.33\ldots \\
27\overline{)2358.00} \\
\underline{216} \\
198 \\
\underline{189} \\
90 \\
\underline{81} \\
90 \\
\underline{81} \\
9\ldots
\end{array}
$$

Hence, the decimal expansion of $\frac{2358}{27}$ is 87.33.... Since the remainder 9 is obtained again and again, the decimal numberis non-terminating and repeating. The decimal number can also be written as $87.\overline{3}$.

Medium

Example 1: Find the decimal expansion of each of the following rational numbers and write the nature of the same.

1. $\mathbf{\frac{65}{101}}$
2. $\mathbf{\frac{923}{400}}$
3. $\mathbf{\frac{37}{99}}$
4. $\mathbf{\frac{67}{100}}$

Solution:

$$\begin{array}{r} 0.643564\ldots \\ \text{i)}\quad 101\overline{)65.000000} \\ \underline{606} \\ 440 \\ \underline{404} \\ 360 \\ \underline{303} \\ 570 \\ \underline{505} \\ 650 \\ \underline{606} \\ 440 \\ \underline{404} \\ 36 \end{array}$$

We have $\frac{65}{101}$ = 0.64356435... = $0.\overline{6435}$

The group of digits '6435' repeats after the decimal point. Hence, the decimal expansion of the given rational number is non-terminating and repeating.

$$\begin{array}{r} 2.3075 \\ \text{ii)}\quad 400\overline{)923.0000} \\ \underline{800} \\ 1230 \\ \underline{1200} \\ 300 \\ \underline{0} \\ 3000 \\ \underline{2800} \\ 2000 \\ \underline{2000} \\ 0 \end{array}$$

We have $\frac{923}{400}$ = 2.3075

Hence, the given rational number has a terminating decimal expansion.

iii)
$$
\begin{array}{r}
0.3737\ldots \\
99\overline{\smash{)}37.0000} \\
\underline{297} \\
730 \\
\underline{693} \\
370 \\
\underline{297} \\
730 \\
\underline{693} \\
37
\end{array}
$$

We have $\frac{37}{99}$ = 0.3737... = $0.\overline{37}$

The pair of digits ‘37’ repeats after the decimal point. Hence, the decimal expansion of the given rational number is non-terminating and repeating.

iv)
$$
\begin{array}{r}
0.67 \\
100\overline{\smash{)}67.00} \\
\underline{600} \\
700 \\
\underline{700} \\
0
\end{array}
$$

We have $\frac{67}{100}$ = 0.67

Hence, the given rational number has a terminating decimal expansion.

Terminating and Non-terminating Repeating Decimal Expansions of Rational Numbers

We can find the decimal expansion of rational numbers using long division method.

However, it is possible to check whether the decimal expansion is terminating or non-terminating without actually carrying out long division also.

Let us start by taking a few rational numbers in the decimal form.

(a)

$$0.5632 = \frac{5632}{10000}$$

On prime factorising the numerator and the denominator, we obtain

$$\frac{5632}{10000} = \frac{2^9 \times 11}{2^4 \times 5^4} = \frac{2^5 \times 11}{5^4}$$

(b)

$$0.275 = \frac{275}{1000}$$

On prime factorizing the numerator and the denominator, we obtain

$$\frac{275}{1000} = \frac{5^2 \times 11}{2^3 \times 5^3} = \frac{11}{2^3 \times 5}$$

Can you see a pattern in the two examples?

We notice that the given examples are rational numbers with terminating decimal expansions. When they are written in the $\frac{p}{q}$ form, where *p* and *q* are co-prime

(the HCF of *p* and *q* is 1), the denominator, when written in the form of prime factors, has 2 or 5 or both.

The above observation brings us to the given theorem.

If x is a rational number with terminating decimal expansion, then it can be expressed in the $\frac{p}{q}$ form, where p and q are co-prime (the HCF of p and q is 1) and the prime factorisation of q is of the form 2n5m, where n and m are non-negative integers.

Contrary to this, if the prime factorisation of q is not of the form 2n5m, where n and m are non-negative integers, then the decimal expansion is a non-terminating one.

Let us see a few examples that will help verify this theorem.

(a) $\frac{7}{12} = \frac{7}{2^2 \times 3} = 0.58333...$

(b) $\frac{15}{16} = \frac{3 \times 5}{2^4} = \frac{3 \times 5 \times 5^4}{2^4 \times 5^4} = \frac{3 \times 5^5}{(10)^4} = 0.9375$

(c) $\frac{1}{14} = \frac{1}{7 \times 2} = 0.0714285714...$

(d) $\frac{125}{16} = \frac{5^3}{2^4} = \frac{5^3 \times 5^4}{2^4 \times 5^4} = \frac{5^7}{(10)^4} = \frac{78125}{10^4} = 7.8125$

Note that in examples **(b)** and **(d)**, each of the denominators is composed only of the prime factors 2 and 5, because of which, the decimal expansion is terminating. However, in examples **(a)** and **(c)**, each of the denominators has at least one prime factor other than 2 and 5 in their prime factorisation, because of which, the decimal expansion is non-terminating and repetitive.

To summarize the above results, we can say that:

Let x = $\frac{p}{q}$ be any rational number.
If the prime factorization of q is of the form 2m5n, where m and n are non-negative integers, then x has a terminating decimal expansion.
If the prime factorisation of q is not of the form 2m5n, where m and n are non-negative integers, then x has a non-terminating and repetitive decimal expansion.

Let us solve a few examples to understand this concept better.

Example 1: Without carrying out the actual division, find if the following rational numbers have a terminating or a non-terminating decimal expansion.

(a) $\frac{17}{1600}$

(b) $\frac{723}{392}$

Solution:

(a) $\frac{17}{1600}=\frac{17}{2^6\times 5^2}$

As the denominator can be written in the form 2^n5^m, where n = 6and m = 2 are non-negative integers, the given rational number has a terminating decimal expansion.

(b) $\frac{723}{392}=\frac{3\times 241}{2^3\times 7^2}$

As denominator cannot be written in the form 2^n5^m, where n and m are non-negative integers, the given rational number has a non-terminating decimal expansion.

Example 2: Without carrying out the actual division, find if the expression $\frac{715}{128}$ has a terminating or a non-terminating decimal expansion.

Solution: $\frac{715}{128}=\frac{715}{2^7}$

As the denominator can be written in the form 2^n5^m, where n = 7and m = 0 are non-negative integers, the given rational number has a terminating decimal expansion.

$$\frac{715}{128}=\frac{715}{2^7}=\frac{715\times 5^7}{2^7\times 5^7}$$
$$=\frac{715\times 5^7}{10^7}$$

$$=\frac{715\times 78125}{10^7}=\frac{55859375}{10^7}=5.5859375$$

Hence, 5.5859375 is the decimal expansion of the given rational number.