

Triangles

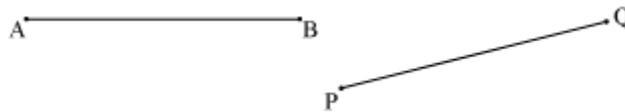
Difference Between Similarity and Congruence

Similar figures and congruent figures may appear to be closely related concepts, but there is an important difference between them.

Congruency of line segments:

“Two line segments are congruent to each other if their lengths are equal”.

Consider the following line segments.



Here, the line segments AB and PQ will be congruent to each other, if they are of equal length.

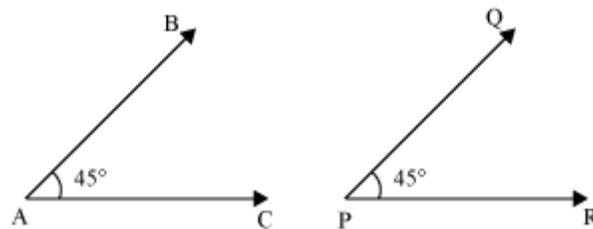
Conversely, we can say that, *“Two line segments are of equal length if they are congruent to each other”.*

i.e. if $\overline{AB} \cong \overline{PQ}$, then $AB = PQ$.

Congruency of angles:

“Two angles are said to be congruent to each other if they have the same measure”.

The angles shown in the following figures are congruent to each other as both the angles are of the same measure 45° .



Thus, we can write $\angle BAC \cong \angle QPR$.

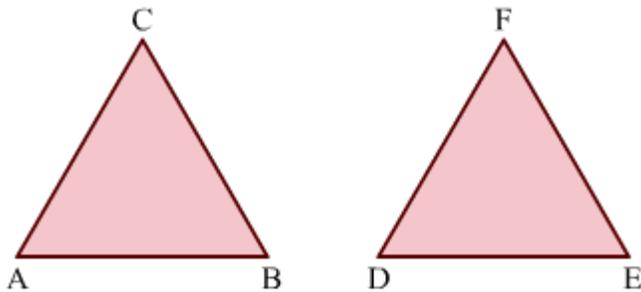
Its converse is also true.

“If two angles are congruent to each other, then their measures are also equal”.

There is one special thing about congruent figures that their corresponding parts are always equal.

For example, if two triangles are congruent then their corresponding sides will be equal. Also, their corresponding angles will be equal.

Look at the following triangles.



Here, $\triangle ABC \cong \triangle DEF$ under the correspondence $\triangle ABC \leftrightarrow \triangle DEF$. This correspondence rule represents that in given triangles, $AB \leftrightarrow DE$ (AB corresponds to DE), $BC \leftrightarrow EF$, $CA \leftrightarrow FD$, $\angle A \leftrightarrow \angle D$, $\angle B \leftrightarrow \angle E$, $\angle C \leftrightarrow \angle F$. These are **corresponding parts of congruent triangles** (CPCT), $\triangle ABC$ and $\triangle DEF$.

Since $\triangle ABC$ and $\triangle DEF$ are congruent, their corresponding parts are equal.

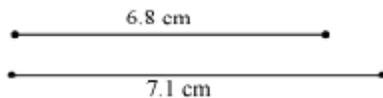
Therefore, $AB = DE$, $BC = EF$, $CA = FD$

And, $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$

Similarly, we can apply the method of CPCT on other congruent triangles also.

Let us now try and apply what we have just learnt in some examples.

Example 1: Find which of the pairs of line segments are congruent.



(i)

(ii)

Solution:

(i) Lengths of the two line segments are not same. Therefore, they are not congruent.

(ii) Each of the line segments is of length 3.1 cm, i.e. they are equal. Therefore, they are congruent.

Example 2: If $\overline{AB} \cong \overline{PQ}$ and $\overline{PQ} = 9$ cm, then find the length of \overline{AB} .

Solution:

Since $\overline{AB} \cong \overline{PQ}$, i.e. line segment AB is congruent to line segment PQ, therefore, \overline{AB} and \overline{PQ} are of equal length.

$$\therefore \overline{AB} = 9 \text{ cm}$$

Example 3: If $\angle ABC \cong \angle PQR$ and $\angle PQR = 75^\circ$, then find the measure of $\angle ABC$.

Solution:

If two angles are congruent, then their measures are equal.

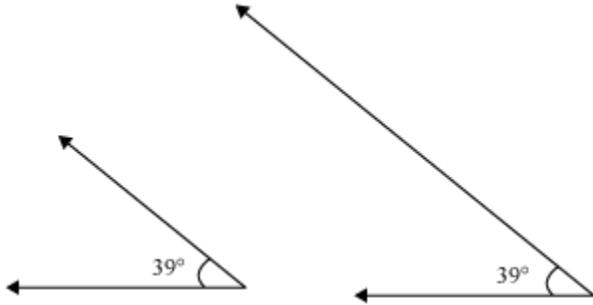
Since $\angle ABC \cong \angle PQR$,

$$\therefore \angle ABC = \angle PQR$$

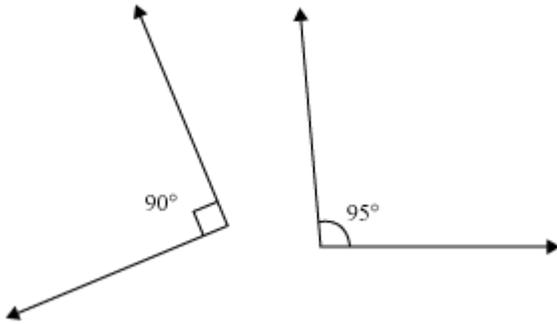
Therefore, $\angle ABC = 75^\circ$

Example 4: Which of the following pairs of angles are congruent?

(i)



(ii)



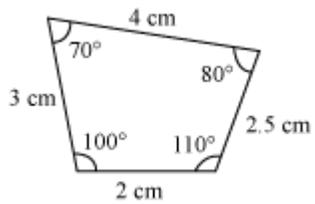
Solution:

(i) The measure of both the angles is the same. Therefore, they are congruent.

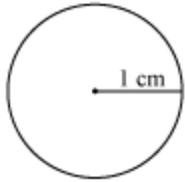
(ii) The measures of the two angles are different. Therefore, they are not congruent.

Example 5: Identify the pairs of similar and congruent figures from the following.

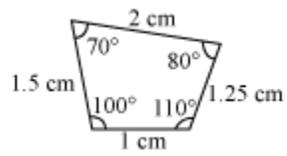
(i)



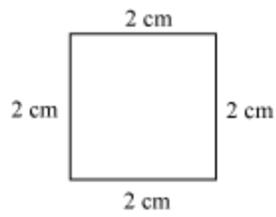
(ii)



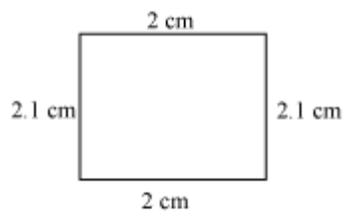
(iii)



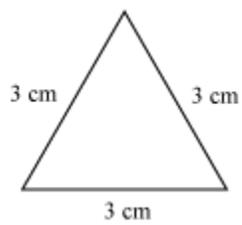
(iv)



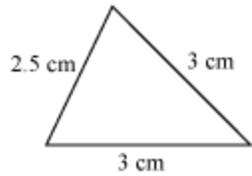
(v)



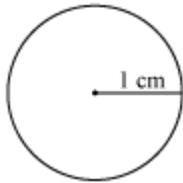
(vi)



(vii)



(viii)



Solution:

Figures (i) and (iii) are similar because their corresponding angles are equal and their corresponding sides are in the same ratio. However, these figures are not congruent as they are of different sizes.

Figures (ii) and (viii) are congruent as they are of the same shape and size (circles with radius 1 cm each).

Example 6:

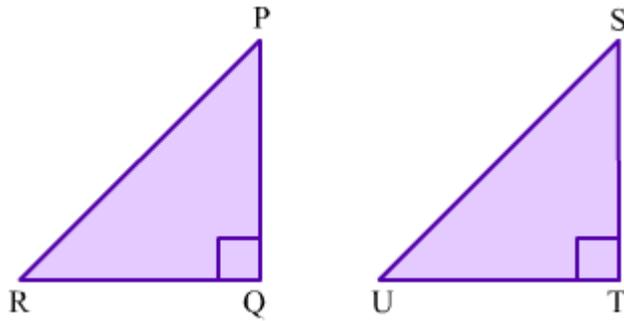
Are the following figures similar or congruent?



Solution:

The two given figures show two one-rupee coins. As both the figures represent the same coin in two different sizes, they are similar to each other. However, the pictures are not congruent because of their different sizes.

Example 7: In the following figure, ΔPQR and ΔSTU are congruent.



If $PQ = 8$ cm, $QR = 6$ cm then find the perimeter of ΔSTU .

Solution:

In ΔPQR , we have

$PQ = 8$ cm, $QR = 6$ cm and $\angle Q = 90^\circ$

Applying Pythagoras theorem in ΔPQR , we obtain

$$RP^2 = PQ^2 + QR^2$$

$$\Rightarrow RP^2 = 8^2 + 6^2$$

$$\Rightarrow RP^2 = 64 + 36$$

$$\Rightarrow RP^2 = 100$$

$$\Rightarrow RP = 10 \text{ cm}$$

Since ΔPQR and ΔSTU are congruent, their corresponding parts will be equal.

Therefore,

$$PQ = 8 \text{ cm} = ST \quad (\text{CPCT})$$

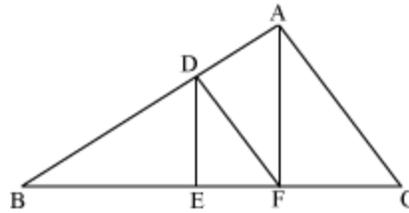
$$QR = 6 \text{ cm} = TU \text{ and} \quad (\text{CPCT})$$

$$RP = 10 \text{ cm} = US \quad (\text{CPCT})$$

$$\therefore \text{Perimeter of } \Delta STU = ST + TU + US = 8 \text{ cm} + 6 \text{ cm} + 10 \text{ cm} = 24 \text{ cm}$$

Basic Proportionality Theorem and Its Converse

Consider the following figure.



In the above figure, DE is parallel to AF and DF is parallel to AC. Can we say that point E divides BF in the same ratio in which point F divides BC?

For this, we have to prove that $\left(\frac{BE}{EF} = \frac{BF}{FC}\right)$.

To prove it, we should have the knowledge of basic proportionality theorem (Thales theorem).

Now, let us solve the problem discussed in the beginning with the help of BPT.

In $\triangle ABF$, we know that AF is parallel to DE.

Thus, using BPT,

$$\frac{BD}{DA} = \frac{BE}{EF} \dots (1)$$

Similarly in $\triangle ABC$, DF is parallel to AC.

Thus, using BPT,

$$\frac{BD}{DA} = \frac{BF}{FC} \dots (2)$$

From equations (1) and (2), we obtain

$$\frac{BE}{EF} = \frac{BF}{FC}$$

Thus, we can say that point E divides BF in the same ratio in which point F divides BC.

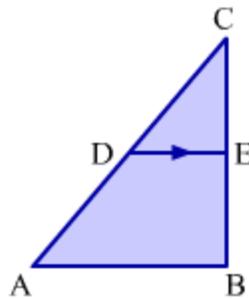
The **converse of BPT** is also true, which can be stated as follows.

“If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side”.

Corollary of BPT:

If a line is drawn parallel to a side of a triangle, then the sides of the new triangle formed are proportional to the sides of the given triangle.

I.e, In the given figure, if $DE \parallel AB$, then $\frac{CD}{CA} = \frac{DE}{AB} = \frac{CE}{CB}$



Applications of Basic Proportionality Theorem:

There are two very important properties based on BPT which are as follows:

1. Property of intercepts made by three parallel lines on a transversal.
2. Property of angle bisector of a triangle.

Let us discuss these properties in detail along with their proofs.

Property 1: Intercept Theorem

The lengths of the intercepts made by three parallel lines on one transversal are in the same ratio as the lengths of the corresponding intercepts made by the same lines on any other transversal.

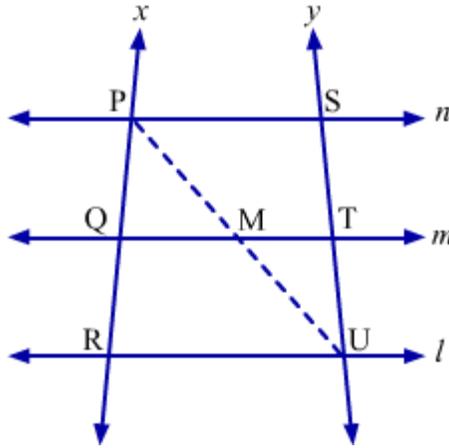
Let us prove this property.

Given: line $l \parallel$ line $m \parallel$ line n

Transversal x intersects these lines at points P, Q and R while transversal y intersects these lines at points S, T and U .

To prove: $\frac{PQ}{QR} = \frac{ST}{TU}$

Construction: Draw a line segment PU intersecting line m at point M .



Proof:

In ΔPRU , we have

$QM \parallel RU$

$$\therefore \frac{PQ}{QR} = \frac{PM}{MU} \quad \dots(1) \quad \dots(\text{By BPT})$$

Similarly, in ΔUSP , we have

$TM \parallel SP$

$$\therefore \frac{UM}{MP} = \frac{UT}{TS} \quad \dots(\text{By BPT})$$

$$\Rightarrow \frac{PM}{MU} = \frac{ST}{TU}$$

$$\Rightarrow \frac{PQ}{QR} = \frac{ST}{TU} \quad [\text{Using (1)}]$$

Hence proved.

Property 2: Angle Bisector Theorem

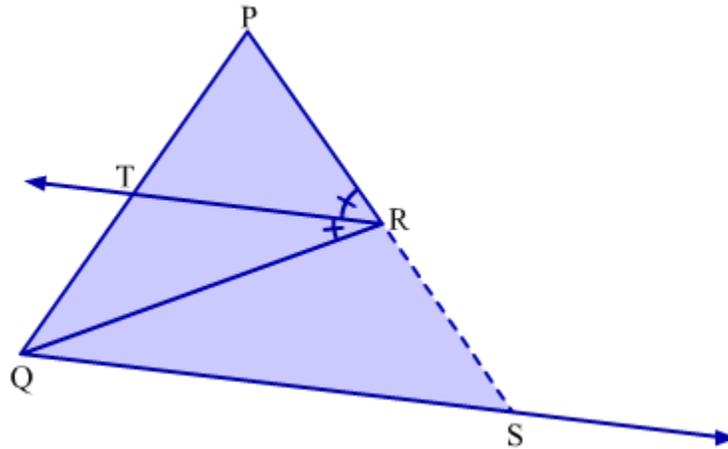
In a triangle, the angle bisector divides the side opposite to the angle in the ratio same as the ratio of remaining sides.

Let us prove this property.

Given: In ΔPQR , ray RT bisects $\angle PRQ$.

To prove: $\frac{PT}{TQ} = \frac{PR}{RQ}$

Construction: Draw a ray from Q parallel to ray RT such that it intersects extended PR at S .



Proof:

We have,

$RT \parallel QS$ and PS is transversal

$$\therefore \angle PRT = \angle RSQ \quad \dots(1) \quad (\text{Corresponding angles})$$

Considering other transversal RQ , we obtain

$$\angle TRQ = \angle RQS \quad \dots(2) \quad (\text{Alternate angles})$$

But $\angle PRT = \angle TRQ$ (RT bisects $\angle PRQ$)

$$\therefore \angle RSQ = \angle RQS \quad [\text{Using (1) and (2)}]$$

Thus, in ΔRQS ,

$$RS = RQ \quad \dots(3) \quad (\text{Side opposite to equal angles are equal})$$

Now, in ΔPQS , we have

$RT \parallel QS$

$$\therefore \frac{PT}{TQ} = \frac{PR}{RS} \quad (\text{By BPT})$$

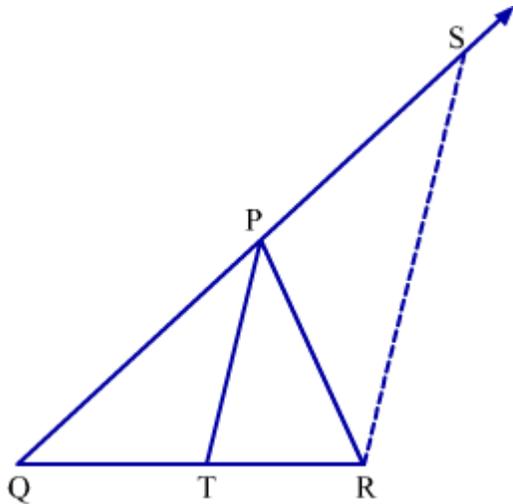
$$\Rightarrow \frac{PT}{TQ} = \frac{PR}{RQ} \quad [\text{Using (3)}]$$

Hence proved.

Property 3: Converse of Angle Bisector Theorem

If a straight line through one vertex of a triangle divides the opposite side in the ratio of the other two sides, then the line bisects the angle at the vertex.

Let us prove this property.



Given: In ΔPQR , line PT divides the opposite side BQ internally such that.

$$\frac{QT}{TR} = \frac{PQ}{PR}$$

To prove: PT bisects $\angle QPR$. i.e. $\angle QPT = \angle RPT$.

Construction: Draw a ray from R parallel to ray PT such that it intersects extended QP at S .

Proof:

Since $PT \parallel SR$, then

$$\frac{QT}{TR} = \frac{QP}{PS} \quad \dots(1) \quad \text{(Basic Proportionality Theorem)}$$

And we have,

$$\frac{QT}{TR} = \frac{QP}{PR} \quad \dots(2) \quad \text{(Given)}$$

From (1) and (2) we have,

$$\frac{QP}{PR} = \frac{QP}{PS}$$

$$\Rightarrow PS = PR$$

$$\text{Now in } \Delta PSR, \angle PSR = \angle PRS \quad \dots(3)$$

If $PT \parallel SR$, then

$$\angle TPR = \angle PRS \quad \dots(4) \quad \text{(Alternate Interior Angles)}$$

$$\angle QPT = \angle PSR \quad \dots(5) \quad \text{(Corresponding Angles)}$$

From (3), (4) and (5) we get

$$\angle QPT = \angle TPR$$

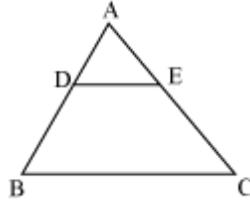
$\therefore PT$ bisect $\angle QPR$.

Hence proved.

These properties are very useful sometimes.

Let us now solve some examples based on BPT, its converse and properties related to BPT.

Example 1: In triangle ABC , D and E are points on the sides AB and AC , such that $AB = 11.2$ cm, $AD = 2.8$ cm, $AC = 14.4$ cm, and $AE = 3.6$ cm. Show that DE is parallel to BC .



Solution:

It is given that,

$AB = 11.2$ cm, $AD = 2.8$ cm, $AC = 14.4$ cm, and $AE = 3.6$ cm

Therefore, $BD = AB - AD = 11.2 - 2.8 = 8.4$ cm

And,

$EC = AC - AE = 14.4 - 3.6 = 10.8$ cm

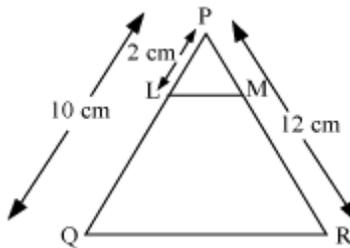
Now,

$$\frac{AD}{DB} = \frac{2.8}{8.4} = \frac{1}{3} \text{ and } \frac{AE}{EC} = \frac{3.6}{10.8} = \frac{1}{3}$$

$$\Rightarrow \frac{AD}{DB} = \frac{AE}{EC}$$

Thus, DE divides sides AB and AC of triangle ABC in the same ratio. Therefore, by the **converse of BPT**, we obtain that DE is parallel to BC.

Example 2: In the figure shown below, find the length of PM, if it is given that $LM \parallel QR$. The corresponding measures are shown in the figure.



Solution:

Here, $LM \parallel QR$

Then, using basic proportionality theorem, we obtain

$$\frac{PL}{LQ} = \frac{PM}{MR} \quad \dots(i)$$

Let $PM = x$ cm

Then, $MR = 12 - x$ cm

And, $PL = 2$ cm

$LQ = 10$ cm $- 2$ cm $= 8$ cm

On putting these values in equation (i), we obtain

$$\frac{2}{8} = \frac{x}{12 - x}$$

$$2(12 - x) = 8x$$

$$24 - 2x = 8x$$

$$24 = 8x + 2x$$

$$24 = 10x$$

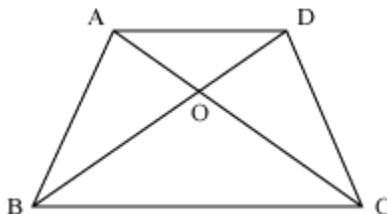
$$x = 2.4$$
 cm

Thus, $PM = 2.4$ cm

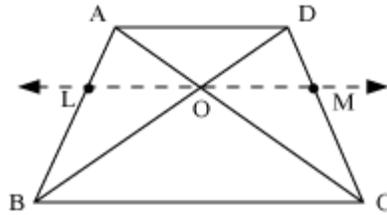
Example 3: If ABCD is a trapezium with $AD \parallel BC$, then prove that $\frac{AO}{CO} = \frac{DO}{BO}$, where O is the point of intersection of diagonals AC and BD.

Solution:

A trapezium has been shown in the following figure.



A line LM is drawn parallel to AD and BC and passing through O.



Here, $LO \parallel BC$.

Using BPT in $\triangle ABC$,

$$\frac{AL}{LB} = \frac{AO}{CO} \quad \dots(i)$$

Similarly, using BPT in $\triangle ABD$ as $LO \parallel AD$, we obtain

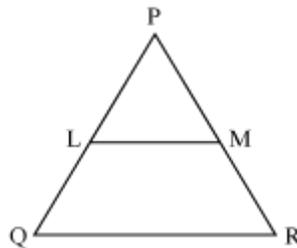
$$\frac{AL}{LB} = \frac{DO}{BO} \quad \dots(ii)$$

From equations (i) and (ii), we obtain

$$\frac{AO}{CO} = \frac{DO}{BO}$$

Hence, proved

Example 4: In $\triangle PQR$, $LM \parallel QR$ and L is the mid-point of side PQ. Show that $PM = MR$.



Solution:

Here, $LM \parallel QR$

Using basic proportionality theorem (BPT),

$$\frac{PL}{LQ} = \frac{PM}{MR} \quad \dots(i)$$

Now, L is the mid-point of PQ.

$$\therefore PL = LQ$$

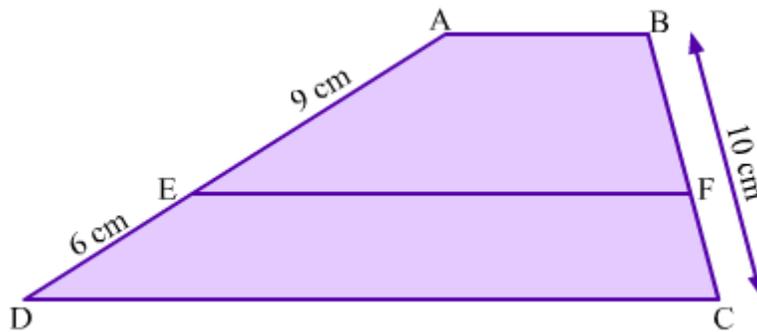
Using this in equation (i), we obtain

$$\begin{aligned} \frac{LQ}{LQ} &= \frac{PM}{MR} \\ \frac{PM}{MR} &= 1 \end{aligned}$$

$$\therefore PM = MR$$

Hence, proved

Example 5: In trapezium ABCD, $AB \parallel EF \parallel DC$. Find the length of BF and FC.



Solution:

In trapezium ABCD, $AB \parallel EF \parallel DC$.

Here, AD and BC are transversals to parallel segments AB, EF and DC.

Intercepts made by AD are AE and ED while intercepts made by BC are BF and FC.

Using property of intercepts made by three parallel lines on a transversal, we obtain

$$\frac{AE}{ED} = \frac{BF}{FC}$$

$$\Rightarrow \frac{9}{6} = \frac{BF}{FC}$$

$$\Rightarrow FC = \frac{6}{9}BF \quad \dots(1)$$

Now,

$$BF + FC = 10$$

$$BF + \frac{6}{9}BF = 10$$

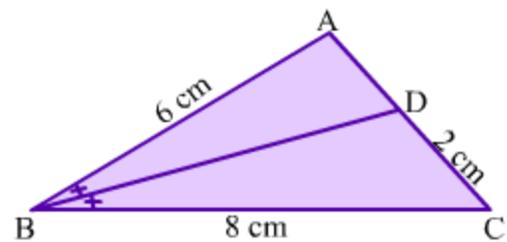
$$\frac{15}{9}BF = 10$$

$$BF = 6$$

$$\therefore FC = \frac{6}{9} \times 6 = 4$$

Thus, $BF = 6$ cm and $FC = 4$ cm.

Example 6: In $\triangle ABC$, BD bisects $\angle ABC$. Find the length of AD .



Solution:

In $\triangle ABC$, BD bisects $\angle ABC$

Thus, by using the property of angle bisector of a triangle, we obtain

$$\frac{AB}{BC} = \frac{AD}{CD}$$

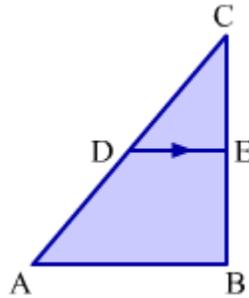
$$\Rightarrow \frac{6}{8} = \frac{AD}{2}$$

$$\Rightarrow AD = 1.5$$

Hence, the length of AD is 1.5 cm.

Example 7.

In the given figure, $DE \parallel AB$. If perimeter of $\triangle ABC$: perimeter of $\triangle CDE = 4:5$ and $DE = 1.2$ cm, then find the length of AB .



Answer:

It is given that, perimeter of $\triangle ABC$: perimeter of $\triangle CDE = 4:5$ and $DE = 1.2$.

In $\triangle ABC$, $DE \parallel AB$.

By applying the corollary of basic proportionality theorem, we get

$$\frac{CD}{CA} = \frac{CE}{CB} = \frac{DE}{AB} = k \text{ (say)}$$

$$CD = kCA, CE = kCB, DE = kAB$$

$$CD + CE + DE = k(CA + CB + AB)$$

$$\text{Perimeter of } \triangle CDE = k \text{Perimeter of } \triangle ABC$$

$$\frac{\text{Perimeter of } \triangle CDE}{\text{Perimeter of } \triangle ABC} = k$$

$$\therefore \frac{CD}{CA} = \frac{CE}{CB} = \frac{DE}{AB} = \frac{\text{Perimeter of } \triangle CDE}{\text{Perimeter of } \triangle ABC}$$

$$\Rightarrow \frac{1.2}{AB} = \frac{5}{4}$$

$$\Rightarrow AB = 1.2 \times \frac{5}{4} = 1.5 \text{ cm}$$

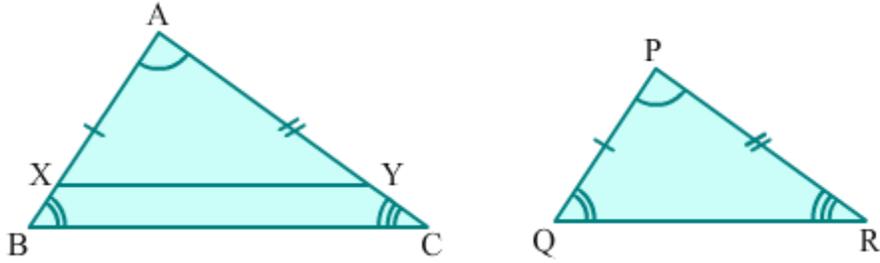
AAA Criterion Of Similarity Of Triangles

We have various criteria to prove two triangles similar. AAA criterion is one of these.

We can check the similarity of any two triangles using AAA criterion of similarity if any two angles of each triangle are given so, AAA criterion is same as AA criterion.

AA criterion "If two triangles are equiangular, then their corresponding sides are proportional." can be proved as below.

Given: $\triangle ABC$ and $\triangle PQR$ where $\angle A = \angle P$, $\angle B = \angle Q$ and $\angle C = \angle R$.



To prove: $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$

Construction: Mark X and Y on AB and AC respectively such that AX = PQ and AY = PR.

Proof:

In $\triangle AXY$ and $\triangle PQR$,

AX = PQ [By construction]

$\angle A = \angle P$ [Given]

AY = PR [By construction]

So, by SAS postulate, $\triangle AXY \cong \triangle PQR$.

[Note: The symbol ' \cong ' stands for congruency]

$\Rightarrow XY = QR$ and $\angle X = \angle Q$ [CPCT]

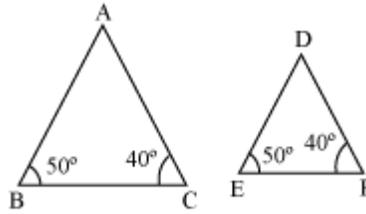
Now, $\angle X = \angle B$ [$\angle X = \angle Q = \angle B$]

$\therefore XY \parallel BC$ [$\angle X$ and $\angle B$ are corresponding angles]

$$\begin{aligned} \therefore \frac{AB}{AX} &= \frac{AC}{AY} = \frac{BC}{XY} \\ \Rightarrow \frac{AB}{PQ} &= \frac{CA}{PR} = \frac{BC}{QR} \end{aligned}$$

Hence, AA criterion is proved.

Now, look at the following triangles.



Here, $\angle B = \angle E = 50^\circ$

and $\angle C = \angle F = 40^\circ$

Then, using AAA similarity criterion, $\triangle ABC$ is similar to $\triangle DEF$.

In symbolic form, we can write $\triangle ABC \sim \triangle DEF$. In symbolic form, the order of vertices is very important. For the above triangles, we cannot write $\triangle ABC \sim \triangle EFD$ because $\angle B = \angle E$ and $\angle C = \angle F$

Let us look at one more application of the AAA similarity criterion in the video in order to get a better understanding of this concept.

Converse of AAA criterion is also true which states that:

If two triangles are similar then their corresponding angles are equal.

For example, if $\triangle ABC \sim \triangle DEF$ then $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

Note: In some state boards, the symbol "|||" is used for similarity.

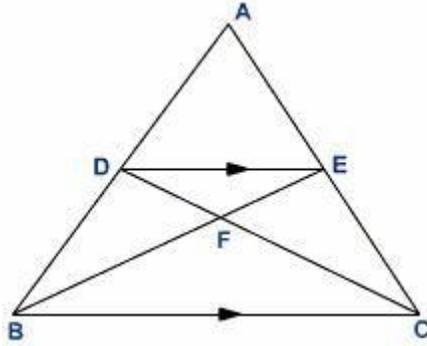
I.e., $\triangle ABC \sim \triangle DEF$ may also be written as $\triangle ABC ||| \triangle DEF$.

Let us now look at some more problems based on AAA similarity criterion.

Example 1: In the following figure, if $DE \parallel BC$, then prove the following.

(a) $\triangle ABC \sim \triangle ADE$

(b) $\triangle DFE \sim \triangle CFB$



Solution:

(a) In $\triangle ABC$ and $\triangle ADE$,

$\angle BAC = \angle DAE$ (Common to both)

$\angle ADE = \angle ABC$ (Since DE is parallel to BC , $\angle ADE$ and $\angle ABC$ are corresponding angles)

By AAA similarity criterion,

$\triangle ABC \sim \triangle ADE$

(b) In $\triangle DFE$ and $\triangle BFC$,

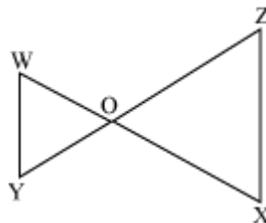
$\angle DFE = \angle BFC$ (Vertically opposite angles)

$\angle EDF = \angle BCF$ (Alternate angles)

By AAA similarity criterion,

$\triangle DFE \sim \triangle CFB$

Example 2: In the given figure, if $WY \parallel ZX$, then prove that $\triangle OWY \cong \triangle OXZ$.



Solution:

Here, $WY \parallel ZX$

Now, in $\triangle OWY$ and $\triangle OZX$,

$\angle WOY = \angle ZOZ$ (Vertically opposite angles)

$\angle OWY = \angle OXZ$ (Alternate angles)

$\angle OYW = \angle OZX$ (Alternate angles)

By AAA similarity criterion of triangles,

$\triangle OWY \sim \triangle OZX$

SSS Criterion of Similarity of Triangles

Now that we have understood the concept of AAA similarity criterion, we will try and understand another similarity criterion which is the SSS similarity criterion. It involves the ratio of the corresponding sides of the two triangles.

Converse of SSS criterion is also true which states that:

If two triangles are similar then their corresponding sides are proportional.

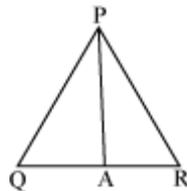
For example, if $\triangle ABC \sim \triangle PQR$ then $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$.

Let us solve some problems to understand this concept better.

Example 1: If $\triangle PQR$ is an isosceles triangle with $PQ = PR$ and A is the mid-point of side QR , then prove that $\triangle PAQ$ is similar to $\triangle PAR$.

Solution:

It is given that $\triangle PQR$ is an isosceles triangle and $PQ = PR$.



In triangles $\triangle PAQ$ and $\triangle PAR$,

$PQ = PR$

Also, A is the mid-point of QR , therefore

$$QA = AR$$

And, $PA = PA$ (Common to both triangles)

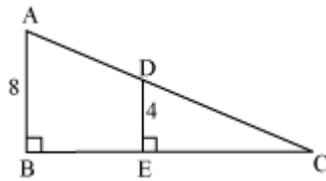
Therefore, we can say that

$$\frac{PQ}{PR} = \frac{QA}{AR} = \frac{PA}{PA}$$

\therefore Using SSS similarity criterion, we obtain

$$\Delta PAQ \sim \Delta PAR$$

Example 2: In the following figure, E and D are the mid-points of the sides BC and AC respectively. Prove that $\Delta ABC \sim \Delta DEC$.



Solution:

It is given that E is the mid-point of BC.

$$\therefore BE = EC$$

$$\text{Now, } BC = BE + EC$$

$$\Rightarrow BC = 2EC$$

$$\Rightarrow \frac{BC}{EC} = \frac{2}{1}$$

Similarly, D is the mid-point of AC, therefore

$$AC = 2DC$$

$$\Rightarrow \frac{AC}{DC} = \frac{2}{1}$$

Also, from the figure,

$$\frac{AB}{DE} = \frac{8}{4} = \frac{2}{1}$$

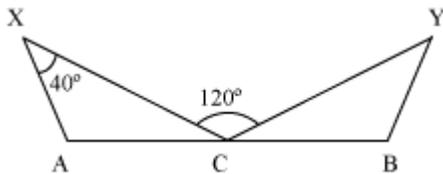
$$\therefore \frac{AB}{DE} = \frac{AC}{DC} = \frac{BC}{EC} = \frac{2}{1}$$

By SSS criterion of similarity of triangles,

$$\triangle ABC \sim \triangle DEC$$

Example 3: In the following figure, the lines XC and YC of same length are drawn such that C is the mid-point of AB. If AX = BY, then find the measure of the following angles.

1. $\angle BYC$ (c) \cong $\angle CAX$
2. $\angle CBY$ (d) \cong $\angle ACX$



Solution:

In the triangles CAX and CBY,

$$CX = CY \text{ (Given)}$$

$$CA = CB \text{ (C is the mid-point of AB)}$$

$$AX = BY \text{ (Given)}$$

Therefore, by SSS similarity criterion,

$$\triangle CAX \sim \triangle CBY$$

We know that the corresponding angles of similar triangles are equal.

$$\therefore \angle AXC = \angle BYC = 40^\circ$$

$$\Rightarrow \angle BYC = 40^\circ$$

Also, $\angle ACX = \angle BCY$

Let $\angle ACX = \angle BCY = x$

Therefore, $x + x + 120^\circ = 180^\circ$ ($\angle ACX$, $\angle BCY$, and $\angle XCY$ form a linear pair)

$$\Rightarrow 2x = 180^\circ - 120^\circ = 60^\circ$$

$$\Rightarrow x = 30^\circ$$

$$\therefore \angle ACX = \angle BCY = 30^\circ$$

Now, by angle sum property in $\triangle ACX$, we obtain

$$30^\circ + \angle CAX + 40^\circ = 180^\circ$$

$$\Rightarrow \angle CAX = 180^\circ - 70^\circ = 110^\circ$$

$$\therefore \angle CBY = \angle CAX = 110^\circ$$

Thus, we obtain

1. $\angle BYC = 40^\circ$
2. $\angle CBY = 110^\circ$
3. $\angle CAX = 110^\circ$
4. $\angle ACX = 30^\circ$

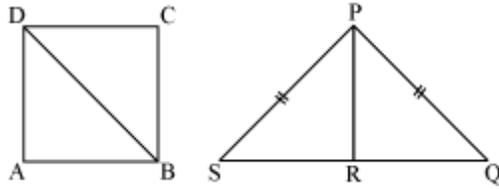
Example 4: ABCD is a square and PQS is an isosceles triangle with $PQ = PS$ and R is the mid-point of QS. If $\triangle ABD \cong \triangle RPQ$, then prove that $\triangle CBD \cong \triangle RPS$.

Solution:

ABCD is a square and PQS is an isosceles triangle.

Therefore, $AB = BC = CD = DA$

And, $PQ = PS$



It is also given that $\triangle ABD \sim \triangle QRP$.

In $\triangle ABD$ and $\triangle CBD$,

$AB = CB$ (Sides of a square)

$BD = BD$ (Common side)

$DA = DC$ (Sides of a square)

By SSS similarity criterion,

$\triangle ABD \sim \triangle CBD$... (2)

Now, in $\triangle RPQ$ and $\triangle RPS$,

$RP = RP$ (Common side)

$PQ = PS$ (Equal sides of an isosceles triangle)

$QR = SR$ (R is the mid-point of QS)

Therefore, $\triangle RPQ \sim \triangle RPS$... (3)

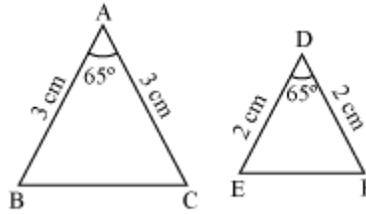
However, $\triangle ABD \sim \triangle RPQ$

Therefore, from (2) and (3), we obtain

$\triangle CBD \sim \triangle RPS$

SAS Criterion Of Similarity Of Triangles

Look at the following figures.



Is there any similarity between them?

We can see that in both the triangles, the lengths of two sides are given and also the measure of the included angle is given. Now, let us compare the sides of the triangles and observe the result we obtain.

On taking the ratio of the sides, we obtain

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{3}{2}$$

Therefore, we observe that the sides of the triangles are in the same ratio i.e., we can say that the sides of the triangles are proportional.

Using the above fact, can we say that the given triangles are similar?

To know the answer, let us first know about a similarity criterion known as SAS similarity criterion.

SAS similarity criterion can be stated as follows.

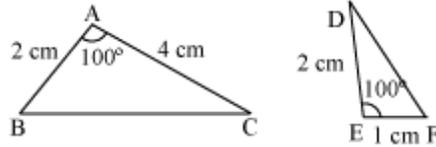
“If one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar”.

Using this criterion, we can check the similarity of any two triangles, if the two sides and the included angle between them are given.

In the above example, $\angle A = \angle D = 65^\circ$ and the sides including these angles are in the same proportion i.e., $\frac{3}{2}$. Thus, we can say that $\triangle ABC$ is similar to $\triangle DEF$.

In symbolic form, we can write $\triangle ABC \sim \triangle DEF$. For writing the symbolic form, the order of the vertices is very important.

For example, consider the following figure.

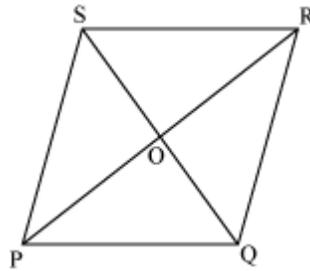


Here, ΔABC and ΔDEF are similar triangles as two sides of both the triangles are proportional and the angles included between them are also equal.

Therefore, we can write $\Delta ABC \sim \Delta EFD$.

Let us now look at some more examples to understand this concept better.

Example1: If PQRS is a parallelogram, then prove that ΔSOR is similar to ΔPOQ .



Solution:

Consider ΔSOR and ΔPOQ .

Since PQRS is a parallelogram, the diagonals bisect each other.

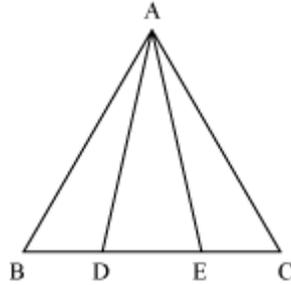
$\therefore SO = OQ$ and $PO = OR$

and $\angle POQ = \angle SOR$ (Vertically opposite angles)

By SAS similarity criterion, we obtain

$\Delta SOR \sim \Delta QOP$

Example2: ΔABC is an isosceles triangle with AB and AC as the equal sides. The points D and E divide the side BC into three equal parts as shown in the figure. Prove that $\Delta ABD \cong \Delta ACE$.



Solution:

Since ABC is an isosceles triangle,

$$AB = AC$$

$\angle ABC = \angle ACB$ (Angles opposite to equal sides are equal in an isosceles triangle)

It is given that the points D and E divide the side BC in three equal parts. Therefore,

$$BD = DE = EC$$

In $\triangle ABD$ and $\triangle AEC$,

$$AB = AC$$

$$BD = EC$$

$$\angle ABD = \angle ACE$$

By SAS similarity criterion,

$$\triangle ABD \sim \triangle AEC$$

Areas Of Similar Triangles

We know what similar triangles are. Now, let us learn about an interesting theorem related to areas of similar triangles.

The theorem states that:

The ratio of areas of two similar triangles is equal to the ratio of squares of their corresponding sides.

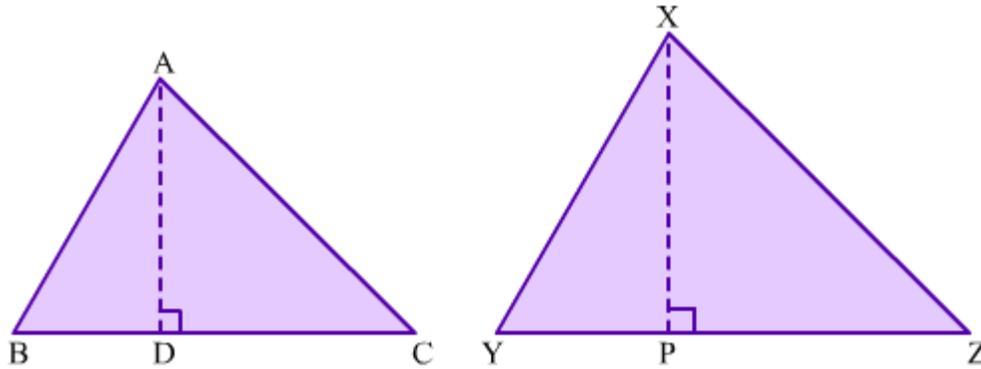
Let us prove this theorem.

Given: $\triangle ABC \sim \triangle XYZ$

To prove: $\frac{A(\triangle ABC)}{A(\triangle XYZ)} = \frac{AB^2}{XY^2} = \frac{BC^2}{YZ^2} = \frac{CA^2}{ZX^2}$

Construction: Draw segment AD perpendicular to BC and segment XP perpendicular to YZ.

Proof:



From the figure, we have

$$A(\triangle ABC) = \frac{1}{2} \times BC \times AD$$

$$\text{And, } A(\triangle XYZ) = \frac{1}{2} \times YZ \times XP$$

$$\therefore \frac{A(\triangle ABC)}{A(\triangle XYZ)} = \frac{BC \times AD}{YZ \times XP} \quad \dots(1)$$

Since $\triangle ABC \sim \triangle XYZ$, we have

$$\angle B = \angle Y, \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX} \quad \dots(2)$$

In $\triangle ADB$ and $\triangle XPY$, we have

$$\angle B = \angle Y$$

$$\angle ADB = \angle XPY \quad (\text{Both are right angles})$$

$$\therefore \triangle ADB \sim \triangle XPY \quad (\text{Using AA similarity test})$$

$$\therefore \frac{AB}{XY} = \frac{AD}{XP} \quad \dots(3)$$

$$\therefore \frac{A(\triangle ABC)}{A(\triangle XYZ)} = \frac{BC}{YZ} \times \frac{AB}{XY} \quad [\text{Using (1) and (3)}]$$

$$\Rightarrow \frac{A(\triangle ABC)}{A(\triangle XYZ)} = \frac{AB}{XY} \times \frac{AB}{XY} \quad [\text{Using (2)}]$$

$$\Rightarrow \frac{A(\triangle ABC)}{A(\triangle XYZ)} = \frac{AB^2}{XY^2}$$

Similarly, it can be shown that

$$\frac{A(\triangle ABC)}{A(\triangle XYZ)} = \frac{BC^2}{YZ^2} \quad \text{and} \quad \frac{A(\triangle ABC)}{A(\triangle XYZ)} = \frac{CA^2}{ZX^2}$$

$$\therefore \frac{A(\triangle ABC)}{A(\triangle XYZ)} = \frac{AB^2}{XY^2} = \frac{BC^2}{YZ^2} = \frac{CA^2}{ZX^2}$$

Thus, the ratio of areas of similar triangles is equal to the ratio of squares of their corresponding sides.

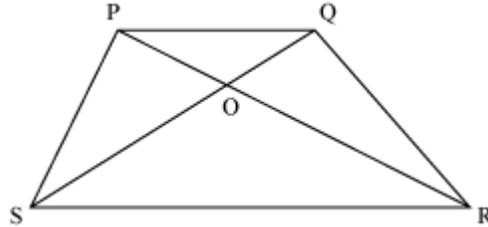
Let A_1 and A_2 be the areas of similar triangles and s_1 and s_2 be their corresponding sides.

Then

$$\frac{A_1}{A_2} = \frac{(s_1)^2}{(s_2)^2} = \left(\frac{s_1}{s_2}\right)^2$$

Now, let us learn to apply this formula with the help of an example.

Consider a trapezium PQRS in which $SR = 3 PQ$. The diagonals PR and QS intersect each other at O.



If the area of ΔPOQ is 9 square cm, then what will be the area of ΔSOR ?

Relation between areas, heights, medians and perimeters of similar triangles:

Let A_1 and A_2 be the areas of two similar triangles such that s_1 and s_2 are their corresponding sides, h_1 and h_2 are their corresponding heights, m_1 and m_2 are their corresponding medians and P_1 and P_2 are their respective perimeters.

Then,

$$\frac{A_1}{A_2} = \frac{(s_1)^2}{(s_2)^2} = \frac{(h_1)^2}{(h_2)^2} = \frac{(m_1)^2}{(m_2)^2} = \frac{(P_1)^2}{(P_2)^2}$$

Let us go through some examples based on the areas of similar triangles.

Example 1: The ratio of areas of two similar triangles is 16:25. Find the ratio of their corresponding sides.

Solution:

We know that,

Ratio of areas of similar triangles = (Ratio of corresponding sides)²

$$\Rightarrow \frac{16}{25} = (\text{Ratio of corresponding sides})^2$$

$$\text{Ratio of corresponding sides} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$= 4:5$$

Example 2: The areas of two similar triangles are 25 cm² and 100 cm². If one side of the first triangle is 4 cm, then find the corresponding side of the other triangle.

Solution:

Let ABC and DEF be two triangles whose areas are 25 cm² and 100 cm² respectively.

Let AB = 4 cm

Then, we have to find DE.

Since the two triangles ABC and DEF are similar,

$$\begin{aligned}\therefore \frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle DEF} &= \frac{(AB)^2}{(DE)^2} \\ \frac{25}{100} &= \frac{(4)^2}{(DE)^2} \\ \frac{1}{4} &= \frac{16}{(DE)^2} \\ (DE)^2 &= 16 \times 4\end{aligned}$$

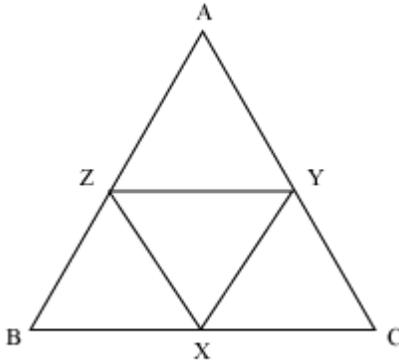
$$(DE)^2 = 64$$

$$DE = 8 \text{ cm}$$

Thus, the corresponding side of the other triangle is 8 cm.

Example 3: In a triangle ABC, X, Y, and Z are the mid-points of the sides BC, AC, and AB respectively. Find the ratios of the areas of $\triangle ABC$ and $\triangle XYZ$.

Solution:



Here, X, Y, and Z are the mid-points of sides BC, AC, and AB respectively.

We know that the line joining the mid-points of two sides is parallel to the third side and its length is half of the third side.

$$\therefore XY \parallel AB \text{ and } XY = \frac{AB}{2}$$

$$YZ \parallel BC \text{ and } YZ = \frac{BC}{2}$$

$$\text{Again, } XZ \parallel AC \text{ and } XZ = \frac{AC}{2}$$

As, $XY \parallel AB$, $YZ \parallel BC$, and $XZ \parallel AC$,

\therefore Quadrilaterals $AYXZ$, $BXYZ$, and $CXZY$ are parallelograms.

$$\therefore \angle BAC = \angle ZXY, \angle ABC = \angle ZYX, \text{ and } \angle ACB = \angle XZY$$

Using AAA similarity criterion, we obtain

$$\Delta ABC \sim \Delta XYZ$$

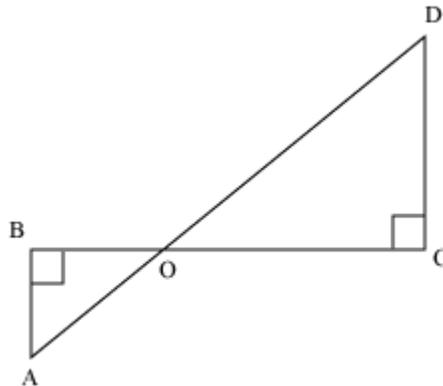
$$\therefore \frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle XYZ} = \left(\frac{AB}{XY} \right)^2$$

$$\frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle XYZ} = \left(\frac{AB}{\left(\frac{AB}{2} \right)} \right)^2$$

$$\frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle XYZ} = \frac{4}{1}$$

Area of $\triangle ABC$: Area of $\triangle XYZ = 4:1$

Example 4: In the given figure, AB and CD are perpendiculars to the line segment BC. Also, AB = 5 cm, CD = 8 cm, and area of $\triangle AOB$ is 175 cm^2 . Find the area of $\triangle COD$.



Solution:

Here, $\triangle AOB$ and $\triangle DOC$ are similar triangles because

$$\angle ABO = \angle DCO \text{ (Each } 90^\circ \text{)}$$

$$\angle AOB = \angle COD \text{ (Vertically opposite angles)}$$

Therefore, by AAA similarity criterion,

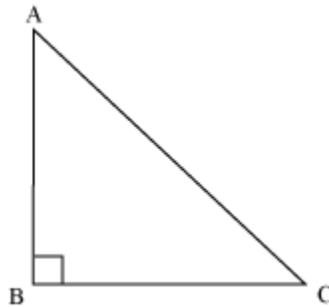
$$\triangle AOB \sim \triangle DOC$$

$$\begin{aligned} \therefore \frac{\text{Area of } \triangle AOB}{\text{Area of } \triangle DOC} &= \frac{(AB)^2}{(CD)^2} \\ \frac{175}{\text{Area of } \triangle DOC} &= \frac{(5)^2}{(8)^2} \\ \text{Area of } \triangle DOC &= \frac{175 \times 64}{25} \\ &= 7 \times 64 \\ &= 448 \text{ cm}^2 \end{aligned}$$

Thus, the area of $\triangle COD$ is 448 cm^2 .

Pythagoras Theorem and Its Applications

Look at the following right triangle ABC.



We have the following relationship between the sides of a right-angled triangle ABC.

$$(AC)^2 = (AB)^2 + (BC)^2$$

This relation between the sides of a right-angled triangle is known as **Pythagoras Theorem**.

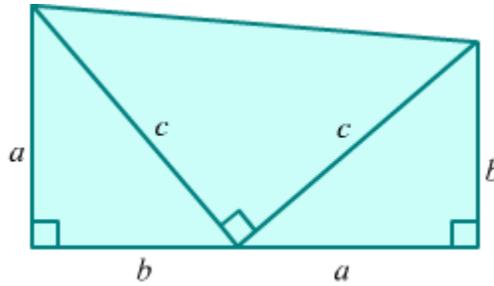
This theorem has several other proofs also. Let us discuss two of them here.

Proof by American President:

The 20th president of United States, James Garfield proved this theorem taking two right angled triangles having sides as a , b and c and an other right angled triangle with side c .

The proof given by him is as follows:

All three triangles are combined as shown in the following figure.



Thus, we got a trapezium.

Area of trapezium = $\frac{1}{2} \times (\text{Sum of lengths of parallel sides}) \times \text{Height}$

$$\Rightarrow \text{Area of trapezium} = \frac{1}{2} \times (a + b)(a + b)$$

$$\Rightarrow \text{Area of trapezium} = \frac{a^2 + 2ab + b^2}{2} \quad \dots(1)$$

And,

$$\text{Sum of areas of 3 triangles} = \frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2} \quad \dots(2)$$

$$\frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2} = \frac{a^2 + 2ab + b^2}{2} \quad [\text{Using (1) and (2)}]$$

$$\Rightarrow \frac{2ab + c^2}{2} = \frac{a^2 + 2ab + b^2}{2}$$

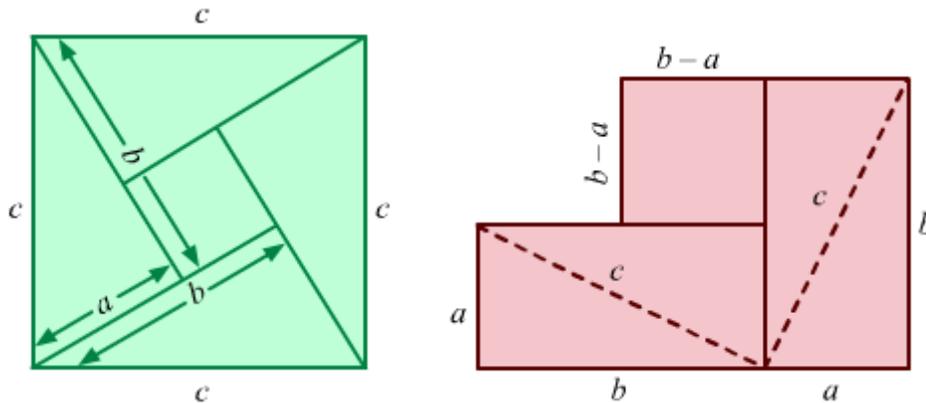
$$\Rightarrow 2ab + c^2 = a^2 + 2ab + b^2$$

$$\Rightarrow c^2 = a^2 + b^2$$

This theorem of right angled triangles was also known to Indian, Chinese, Greek and Babylonian mathematician long before Pythagoras lived. Thus, it was proved by the mathematicians of that time differently.

Proof by Indian Mathematician:

Bhaskaracharya, the great Indian mathematician of 2nd century AD, used the below given diagrams to prove this theorem.



$$\text{Area of 4 triangles} = 4\left(\frac{1}{2}ab\right) = 2ab$$

$$\text{Area of small square} = (b - a)^2 = b^2 - 2ab + a^2$$

Area of big square = Area of 4 triangles + Area of small square

$$\Rightarrow c^2 = 2ab + (b^2 - 2ab + a^2)$$

$$\Rightarrow c^2 = 2ab + b^2 - 2ab + a^2$$

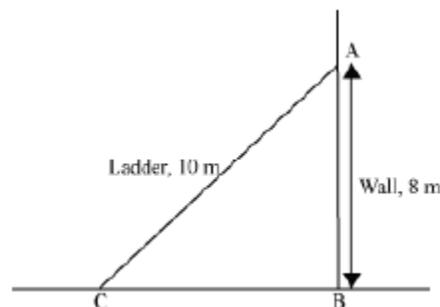
$$\Rightarrow c^2 = b^2 + a^2$$

Bhaskaracharya proved this theorem by using property of similarity also, but we will not discuss it here.

Note: Conventionally in ΔABC , we consider the lengths of sides opposite to vertices A, B and C as a , b and c respectively.

In real life, we come across many situations where a right angle is formed. Let us consider such a situation.

A 10m long ladder is placed on a wall such that the ladder touches the wall at 8m above the ground. This situation can be shown geometrically as follows.



In the above figure, AB is the wall of height 8 m and AC is the ladder of length 10 m. We know that a wall is perpendicular to the floor, i.e. AB is perpendicular to BC. Thus, $\angle ABC$ is a right angle.

Now, can we calculate the distance of the foot of the ladder from the base of the wall?

We can calculate the distance of the foot of the ladder from the base of the wall by using Pythagoras theorem.

In this way, we can use Pythagoras theorem in many situations where right-angled triangle is formed.

Is the converse of Pythagoras theorem also true?

Yes, the converse of Pythagoras theorem is also true.

Its converse can be stated as follows:

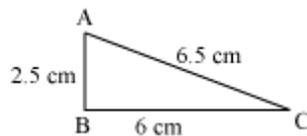
“In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle”.

But how will we prove it?

Thus, in a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.

Using the converse, we can check whether the given triangle is a right triangle or not.

Let ABC be a triangle with sides $AB = 2.5$ cm, $BC = 6$ cm and $CA = 6.5$ cm. Can we say that the triangle ABC is a right-angled triangle?



Here, $(AB)^2 = (2.5)^2 = 6.25$

$(BC)^2 = 6^2 = 36$

And, $(CA)^2 = (6.5)^2 = 42.25$

Therefore, we obtain

$$(AB)^2 + (BC)^2 = 6.25 + 36 = 42.25 = (CA)^2$$

Thus, using the converse of Pythagoras theorem, we can say that the angle opposite to the side CA, i.e. $\angle B$, is a right angle.

Pythagorean triplet: Any three natural numbers a, b, c form a Pythagorean triplet if it satisfies $a^2 = b^2 + c^2$ irrespective of order.

General form to find the Pythagorean Triplets:

For natural number: $2n, (n^2 - 1), (n^2 + 1)$ where n may be even or odd.

For odd natural numbers: $n, \frac{1}{2}(n^2 - 1), \frac{1}{2}(n^2 + 1)$ where n is odd, $n \in N$.

Any number of pythagorean triplets can be generated by giving values to n .

For example, the number 39, 80, and 89 forms a Pythagorean triplet.

$$39^2 = 1521$$

$$80^2 = 6400$$

$$89^2 = 7921$$

$$\text{Now, } 1521 + 6400 = 7921$$

$$\therefore 39^2 + 80^2 = 89^2$$

Here, the square of a number is equal to the sum of the squares of the other two numbers. Therefore, we can say that 39, 80, and 89 forms a Pythagorean triplet.

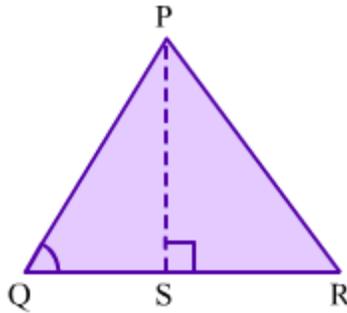
Applications of Pythagoras theorem:

Though Pythagoras theorem holds true for only right angled triangle, it can be applied to obtuse and acute angled triangles as well. The obtained results are discussed below:

(1) In acute angled ΔPQR , if $PS \perp QR$, QSR is a line and $\angle Q < 90^\circ$ then

$$PR^2 = PQ^2 + QR^2 - 2QR \cdot QS$$

Proof: Observe the acute angled triangle ΔPQR .



Here, $PS \perp QR$, QSR is a line and $\angle Q < 90^\circ$.

In $\triangle PSQ$, we have

$$PQ^2 = PS^2 + QS^2 \quad \dots(1) \quad (\text{By Pythagoras theorem})$$

In $\triangle PSR$, we have

$$PR^2 = PS^2 + SR^2 \quad (\text{By Pythagoras theorem})$$

$$\Rightarrow PR^2 = PS^2 + (QR - QS)^2$$

$$\Rightarrow PR^2 = PS^2 + QR^2 + QS^2 - 2QR \cdot QS$$

$$\Rightarrow PR^2 = (PS^2 + QS^2) + QR^2 - 2QR \cdot QS$$

$$\Rightarrow PR^2 = PQ^2 + QR^2 - 2QR \cdot QS \quad [\text{Using (1)}]$$

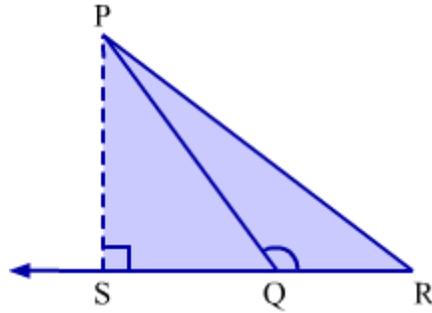
Hence proved.

(2) In obtuse angled $\triangle PQR$, if $PS \perp QR$, QSR is a line and $\angle Q > 90^\circ$ then

$$PR^2 = PQ^2 + QR^2 + 2QR \cdot QS$$

Proof:

Observe the acute angled triangle $\triangle PQR$.



Here, $PS \perp QR$, SQR is a line and $\angle Q > 90^\circ$.

In ΔPSQ , we have

$$PQ^2 = PS^2 + QS^2 \quad \dots(1) \quad (\text{By Pythagoras theorem})$$

In ΔPSR , we have

$$PR^2 = PS^2 + SR^2 \quad (\text{By Pythagoras theorem})$$

$$\Rightarrow PR^2 = PS^2 + (QR + QS)^2$$

$$\Rightarrow PR^2 = PS^2 + QR^2 + QS^2 + 2QR \cdot QS$$

$$\Rightarrow PR^2 = (PS^2 + QS^2) + QR^2 + 2QR \cdot QS$$

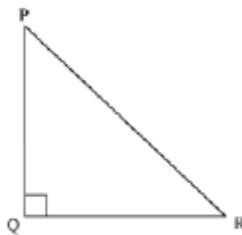
$$\Rightarrow PR^2 = PQ^2 + QR^2 + 2QR \cdot QS \quad [\text{Using (1)}]$$

Hence proved.

Now, let us discuss some more examples based on Pythagoras theorem and its converse.

Example 1: ΔPQR is an isosceles triangle, right angled at Q . Prove that $PR^2 = 2PQ^2$.

Solution:



Here, ΔPQR is an isosceles triangle, right angled at Q . Therefore,

$$PQ = QR \dots (1)$$

Now, using Pythagoras theorem, we obtain

$$(PR)^2 = (PQ)^2 + (QR)^2$$

Using equation (1), we obtain

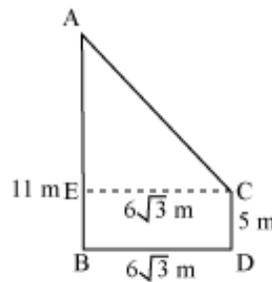
$$(PR)^2 = (PQ)^2 + (PQ)^2$$

$$(PR)^2 = 2(PQ)^2$$

Example 2: Two poles are of length 5m and 11m. The distance between the feet of the poles is $6\sqrt{3}$ m. Find the distance between the tops of the poles.

Solution:

The figure for the given situation can be drawn as follows.



In the above figure, the poles are denoted by AB and CD, where AB = 11 m and CD = 5 m. The distance between the feet of the poles, i.e. BD, is $6\sqrt{3}$ m.

Let us draw a perpendicular CE from C on AB.

AC is the distance between the top of the poles.

$$\text{Here, } BD = CE = 6\sqrt{3} \text{ m}$$

$$\text{And } AE = AB - BE$$

$$= AB - CD \text{ [since } BE = CD]$$

$$= (11 - 5) \text{ m}$$

$$= 6 \text{ m}$$

Using Pythagoras theorem in ΔACE , we obtain

$$(AC)^2 = (AE)^2 + (EC)^2$$

$$(AC)^2 = (6)^2 \text{ m}^2 + (6\sqrt{3})^2 \text{ m}^2$$

$$= (36 + 108) \text{ m}^2$$

$$= 144 \text{ m}^2$$

$$\Rightarrow AC = \sqrt{144 \text{ m}^2}$$

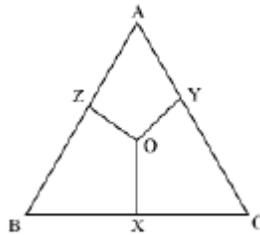
$$\Rightarrow AC = 12 \text{ m}$$

Thus, the distance between the tops of the poles is 12 m.

Example 3: O is any point in the interior of ΔABC and OX, OY, and OZ are the perpendiculars drawn from O to BC, CA, and AB respectively.

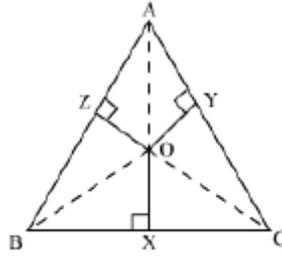
Prove that

$$AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2$$



Solution:

Let us join OA, OB, and OC in the given figure.



Using Pythagoras theorem in ΔOZA , we obtain

$$OA^2 = OZ^2 + AZ^2$$

$$\text{or } AZ^2 = OA^2 - OZ^2 \dots \text{(i)}$$

Similarly, in ΔBOX and ΔCOY respectively, we obtain

$$BX^2 = OB^2 - OX^2 \dots \text{(ii)}$$

$$\text{and } CY^2 = OC^2 - OY^2 \dots \text{(iii)}$$

On adding equations (i), (ii) and (iii), we obtain

$$AZ^2 + BX^2 + CY^2 = OA^2 + OB^2 + OC^2 - OZ^2 - OX^2 - OY^2$$

$$AZ^2 + BX^2 + CY^2 = (OA^2 - OY^2) + (OB^2 - OZ^2) + (OC^2 - OX^2) \dots \text{(iv)}$$

Using Pythagoras theorem in ΔOYA , we obtain

$$OA^2 = OY^2 + AY^2$$

$$\text{Or } AY^2 = OA^2 - OY^2$$

$$\text{Similarly, } BZ^2 = OB^2 - OZ^2$$

$$\text{and } CX^2 = OC^2 - OX^2$$

Using these in equation (iv), we obtain

$$AZ^2 + BX^2 + CY^2 = AY^2 + BZ^2 + CX^2$$

Hence, proved.