Probability

Theoretical and Experimental Probability

Consider an experiment of tossing a coin. Before tossing a coin, we are not sure whether head or tail will come up. To measure this uncertainty, we will find the probability of getting a head and the probability of getting a tail.

A student tosses a coin 1000 times out of which 520 times head comes up and 480 times tail comes up.

The probability of getting a head is the ratio of the number of times head comes up to the total number of times he tosses the coin.

Probability of getting a head $= \frac{520}{1000}$

= 0.52

Similarly, probability of getting a tail $= \frac{480}{1000}$

= 0.48

These are the probabilities obtained from the result of an experiment when we actually perform the experiment. The probabilities that we found above are called **experimental (or empirical) probabilities.**

On the other hand, the probability we find through the theoretical approach without actually performing the experiment is called theoretical probability.

The **theoretical probability (or classical probability)** of an event E, is denoted by P(E) and is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}}$$

Here, we assume that the outcomes of the experiment are equally likely.

When a coin is tossed, there are two possible outcomes. We can either get a head or a tail and these two outcomes are equally likely. The chance of getting a head or a tail is 1.

Thus, probability of getting a head P(E)

$$= \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}} = \frac{1}{2}$$

Similarly, probability of getting a tail = $\frac{1}{2}$

Here, $\frac{1}{2}$ (or 0.5) is the theoretical probability.

Relation between Experimental and Theoretical Probabilities:

There is a fact that the experimental probability may or may not be equal to the theoretical probability.
For example, if we take a coin and toss it by a particular number of times then the theoretical probability of getting a head or a tail will be $\frac{1}{2} = 0.5$ in each trial, but if we observe the outcomes of all the trials and calculate the experimental probability for head or tail then it will not be exactly equal to theoretical probability.

For approval, we can consider the theoretical and experimental probabilities of getting tail in the above experiment.

We have:
Theoretical probability of getting head = 0.52

Also, in our experiment we obtained:

Experimental probability of getting tail = 0.5

It can be observed that experimental probability is not exactly equal to theoretical probability, but very close to it.

Also, experimental probability of the same event can vary according to the number of trials.

Probability

Consider an experiment where two coins are tossed simultaneously. So, can you find out the probability of getting at least one head?

In this experiment, we are not actually performing the experiment. Therefore, here we can find the probability of getting at least one head using theoretical approach.

The theoretical probability for an event E can be defined as follows.

$$P(E) = \frac{\text{Number of outcomes favourable to event } E}{\text{Number of all possible outcomes}}$$

Using this formula, let us try to find the probability of getting at least one head.

The numbers of possible outcomes by tossing two coins simultaneously are as follows.

HH, HT, TH, TT

∴ Total number of outcomes = 4

The favourable outcomes for getting at least one head are

HH, HT, TH

∴ Number of favourable outcomes = 3

Probability of getting at least one head = $\frac{3}{4}$

= 0.75

Let us consider some more examples and calculate the probability using theoretical approach.

Example 1: A dice is thrown and the number on the upper face of the dice is noted. Find the probability of getting

(a) 3

(b) 7

(c) an odd number

(d) an even number

(e) a number less than or equal to 5

(f) a number greater than 2.

Solution:

When a dice is thrown, any of the numbers 1, 2, 3, 4, 5, and 6 can be on the upper face of the dice.

∴ Total number of outcomes = 6

(a) Getting 3

Here, the favourable outcome is only 3.

∴ Number of favourable outcomes = 1

∴ Probability of getting a number 3 on the upper face = $\frac{1}{6}$

(b) Getting 7

There is no favourable outcome as we cannot get 7 by throwing a dice.

∴ Number of favourable outcomes = 0

∴ Probability of getting 7 on the upper face of the dice = $\frac{0}{6} = 0$

(c) Getting an odd number

The favourable outcomes are 1, 3, and 5.

∴ Number of favourable outcomes = 3

∴ Probability of getting an odd number = $\frac{3}{6} = \frac{1}{2}$

(d) Getting an even number

The favourable outcomes are 2, 4, and 6.

∴ Number of favourable outcomes = 3

∴ Probability of getting an even number = $\frac{3}{6}=\frac{1}{2}$

(e) Getting a number less than equal to 5

The favourable outcomes are 1, 2, 3, 4, and 5.

∴ Number of favourable outcomes = 5

∴ Probability of getting a number less than equal to 5 = $\frac{5}{6}$

(f) Getting a number greater than 2

The favourable outcomes are 3, 4, 5, and 6.

∴ Number of favourable outcomes = 4

∴ Probability of getting a number greater than 2 = $\frac{4}{6}=\frac{2}{3}$

Example 2: In tossing three coins simultaneously, find the probability of getting

(a) at least one head

(b) at most one tail

(c) at most two heads

(d) exactly two heads

(e) at least one head but at most one tail

(f) no head at all

(g) at least one head but at most two heads

Solution:

In tossing three coins, the possible outcomes are HHH, HTT, THT, TTH, HHT, HTH, THH, and TTT.

∴ Number of all possible outcomes = 8

(a) Let E be the event of getting at least one head.

The outcomes favourable to E are HHH, HTT, THT, TTH, HHT, HTH, and THH.

∴ Number of outcomes favourable to E = 7

$$\therefore P(E)=\frac{7}{8}$$

(b) Let E be the event of getting at most one tail.

The outcomes favourable to event E are HHH, HHT, HTH, and THH.

∴ Number of outcomes favourable to E = 4

$$\therefore P(E)=\frac{4}{8}=\frac{1}{2}$$

(c) Let E be the event of getting at most two heads.

The outcomes favourable to E are HTT, THT, TTH, HHT, HTH, THH and TTT.

∴ Number of outcomes favourable to E = 7

$$\therefore P(E)=\frac{7}{8}$$

(d) Let E be the event of getting exactly two heads.

The outcomes favourable to E are HHT, HTH, and THH.

∴ Number of outcomes favourable to E = 3

$$\therefore P(E)=\frac{3}{8}$$

(e) Let E be the event of getting at least one head but at most one tail.

The outcomes favourable to E are HHH, HHT, HTH and THH

∴ Number of outcomes favourable to E = 4

$\therefore P(E)=\frac{4}{8}=\frac{1}{2}$

(f) Let E be the event of getting no head at all.

The outcome favourable to E is TTT.

∴ Number of outcomes favourable to E = 1

$\therefore P(E)=\frac{1}{8}$

(g) Let E be the event of getting at least one head but at most two heads.

The outcomes favourable to E are HTT, THT, TTH, HHT, HTH, and THH.

∴ Number of outcomes favourable to E = 6

$\therefore P(E)=\frac{6}{8}=\frac{3}{4}$

Example 3: In the experiment of throwing two dices, find the probability that the sum of two numbers appearing on the upper face is

(i) 7

(ii) greater than 11

Solution:

Let E denote the event of getting a sum 7 and F denote the event of getting a sum greater than 11.

The possible outcomes of this experiment have been shown in the following table.

	1	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1,5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3,5)	(3, 6)

4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4,5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5,5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6,5)	(6, 6)

Number of all possible outcomes = 36

(i) The outcomes favourable to E are (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), and (6, 1).

Number of outcomes favourable to event E = 6

$\therefore P(E)=\frac{6}{36}=\frac{1}{6}$

(ii) The outcome favourable to F is (6, 6).

Number of outcomes favourable to event F = 1

$\therefore P(F)=\frac{1}{36}$

Example 4: 8 defective bulbs are accidentally mixed with 92 good bulbs. When one bulb is drawn at random, it is not possible to look at the bulb and say whether it is defective or non defective. What is the probability that the bulb drawn is non-defective?

Solution:

Let E be the event of drawing a non defective bulb.

Total number of bulbs = 92 + 8

= 100

Number of non defective bulbs = 92

$\therefore P(E)=\frac{92}{100}=0.92$

Example 5: From a pack of cards, five of club and spade, jack of heart, ten of diamond and spade, and queen of club is put aside and then the remaining cards are well shuffled. Find the probability of getting

(a) a card of red queen

(b) a card of black queen

(c) a ten

(d) a five of club

(e) a jack

(f) an ace

(g) a face card

(h) a black face card

Solution:

After taking out 6 cards (five of club and spade, jack of heart, ten of diamond and spade, and queen of club) from 52 cards, we are left with 46 cards.

∴ Number of possible outcomes = 46

(a) Let *E* be the event of getting a card of red queen.

Number of outcomes favourable to *E* = 2 (A queen of diamond and a queen of heart)

∴ $P(E)=\frac{2}{46}=\frac{1}{23}$

(b) Let *E* be the event of getting a card of black queen.

Number of outcomes favourable to *E* = 1 (Already queen of club is taken out)

∴ $P(E)=\frac{1}{46}$

(c) Let *E* be the event of getting a card of ten.

Number of outcomes favourable to *E* = 2 (Ten of heart and club)

∴ $P(E)=\frac{2}{46}=\frac{1}{23}$

(d) Let *E* be the event of getting a five of club.

Number of outcomes favourable to *E* = 0 (Already five of club is taken out)

∴ $P(E)=\frac{0}{46}=0$

(e) Let *E* be the event of getting a jack.

Number of outcomes favourable to *E* = 3 (Already jack of heart is taken out)

∴ $P(E)=\frac{3}{46}$

(f) Let *E* be the event of getting an ace.

Number of outcomes favourable to *E* = 4

∴ $P(E)=\frac{4}{46}=\frac{2}{23}$

(g) Let *E* be the event of getting a face card.

Number of outcomes favourable to *E* = 10 (Already two face cards have been removed)

∴ $P(E)=\frac{10}{46}=\frac{5}{23}$

(h) Let *E* be the event of getting a black face card.

Number of outcomes favourable to *E* = 5 (Already queen of club has been removed)

∴ $P(E)=\frac{5}{46}$

Example 6: A bag contains 25 balls of red and green colour. If 3 red balls and 2 green balls are added, then the probability of getting green is twice the probability of getting red. Find the number of balls of each colour.

Solution:

Total numbers of balls in the bag = 25

Let the bag contain *x* number of red balls.

∴ Number of green balls = 25 − *x*

By adding 3 red balls and 2 green balls,

Total numbers of balls in the bag = 25 + 2 + 3 = 30

Number of red balls = *x* + 3

∴ Number of green balls = 25 − *x* + 2= 27 − x

The probability of ball of each colour is

$P(\text{red balls})=\frac{x+3}{30}$ and $P(\text{green balls})=\frac{27-x}{30}$

But it is given that

$P(\text{green balls})$ = 2 × $P(\text{red balls})$

$\frac{27-x}{30}$ = 2 × $\frac{x+3}{30}$

27 − *x* = 2*x* + 6

3*x* = 21

x = 7 (Number of red balls)

∴ Number of green balls = 25 − *x* = 25 − 7 = 18

Thus, the bag contains 7 red balls and 18 green balls.

Example 7: 25 cards numbered 6, 7, 8, 9, 10 … 30 are put in a box and mixed thoroughly. If one card is drawn at random, then find the probability that the number on the card is

(a) odd

(b) even

(c) prime

(d) divisible by 5

(e) divisible by both 2 and 5

Solution:

We have, total number of cards = 25

∴ Total number of outcomes = 25

(a) Let *E* be the event of getting an odd number.

The favourable outcomes are 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, and 29.

∴ Number of outcomes favourable to *E* = 12

∴ $P(E)=\frac{12}{25}$

(b) Let *X* be the event of getting an even number.

The favourable outcomes are 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, and 30.

∴ Number of outcomes favourable to *X* = 13

∴ $P(X)=\frac{13}{25}$

(c) Let *F* be the event of getting a prime number.

The favourable outcomes are 7, 11, 13, 17, 19, 23, and 29.

∴ Number of outcomes favourable to *F* = 7

∴ $P(F)=\frac{7}{25}$

(d) Let *Y* be the event of getting a number divisible by 5.

The favourable outcomes are 10, 15, 20, 25, and 30.

∴ Number of outcomes favourable to Y = 5

∴ $P(F) = \frac{5}{25} = \frac{1}{5}$

(e) Let G be the event of getting a number divisible by both 2 and 5 (i.e. getting a number divisible by 10).

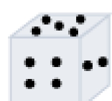
The favourable outcomes are 10, 20, and 30.

∴ Number of outcomes favourable to G = 3

∴ $P(F) = \frac{3}{25}$

Finding Probability Using Complement of a Known Event

Consider the experiment of throwing a dice. Any of the numbers 1, 2, 3, 4, 5, or 6 can come up on the upper face of the dice. We can easily find the probability of getting a number 5 on the upper face of the dice?



Mathematically, probability of any event E can be defined as follows.

$$P(E) = \frac{\text{Number of outcomes favourable to event } E}{\text{Number of all possible outcomes}} = \frac{n(E)}{n(S)}$$

Here, S represents the sample space and $n(S)$ represents the number of outcomes in the sample space.

For this experiment, we have

Sample space (S) = {1, 2, 3, 4, 5, 6}. Thus, S is a finite set.

So, we can say that the possible outcomes of this experiment are 1, 2, 3, 4, 5, and 6.

∴ Number of all possible outcomes = 6

Number of favourable outcomes of getting the number 5 = 1

∴ Probability (getting 5) $= \frac{1}{6}$

Similarly, we can find the probability of getting other numbers also.

P (getting 1) $= \frac{1}{6}$, *P* (getting 2) $= \frac{1}{6}$, *P* (getting 3) $= \frac{1}{6}$, *P* (getting 4) $= \frac{1}{6}$ and

P (getting 6) $= \frac{1}{6}$

Let us add the probability of each separate observation.

This will give us the sum of the probabilities of all possible outcomes.

P (getting 1) + *P* (getting 2) + *P* (getting 3) + *P* (getting 4) + *P* (getting 5) + *P* (getting 6) = $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ = 1

∴ "**Sum of the probabilities of all elementary events is 1**".

Now, let us find the probability of **not** getting 5 on the upper face.

The outcomes favourable to this event are 1, 2, 3, 4, and 6.

∴ Number of favourable outcomes = 5

∴ P (not getting 5) $= \frac{5}{6}$

We can also see that *P* (getting 5) + *P* (not getting 5) $= \frac{1}{6} + \frac{5}{6} = 1$

∴ "**Sum of probabilities of occurrence and non occurrence of an event is 1**".

i.e. **If *E* is the event, then *P* (*E*) + *P* (not *E*) = 1** … (1)

or we can write ***P(E) = 1 − P (*not *E*)**

Here, the events of getting a number 5 and not getting 5 are complements of each other as we cannot find an observation which is common to the two observations.

Thus, event **not *E*** is the complement of event *E.* Complement of event *E* is denoted by $\bar{E}$ or E'.

Using equation (1), we can write

$$\boldsymbol{P(E) + P(\bar{E}) = 1}$$

or

$$\boldsymbol{P(\bar{E}) = 1 - P(E)}$$

This is a very important property about the probability of complement of an event and it is stated as follows:

If *E* is an event of finite sample space S, then $P(\bar{E}) = 1 - P(E)$ where $\bar{E}$ is the complement of event *E*.

Now, let us prove this property algebraically.

Proof:

We have,

$E \cup \bar{E} = S$ and $E \cap \bar{E} = \phi$

$\Rightarrow n(E \cup \bar{E}) = n(S)$ and $n(E \cap \bar{E}) = n(\phi)$

$\Rightarrow n(E \cup \bar{E}) = n(S)$ and $n(E \cap \bar{E}) = 0$ ...(1)

Now,

$n(E \cup \bar{E}) = n(S)$

$\Rightarrow n(E) + n(\bar{E}) - n(E \cap \bar{E}) = n(S)$

$\Rightarrow n(E) + n(\bar{E}) - 0 = n(S)$ [Using (1)]

$\Rightarrow n(\bar{E}) = n(S) - n(E)$

On dividing both sides by $n(S)$, we get

$$\frac{n(\bar{E})}{n(S)} = \frac{n(S)}{n(S)} - \frac{n(E)}{n(S)}$$

 $P(\bar{E}) = 1 - P(E)$

Hence proved.

Let us solve some examples based on this concept.

Example 1: One card is drawn from a well shuffled deck. What is the probability that the card will be

(i) a king?

(ii) not a king?

Solution:

Let *E* be the event 'the card is a king' and *F* be the event 'the card is not a king'.

(i) Since there are 4 kings in a deck.

∴ Number of outcomes favourable to $E = 4$

Number of possible outcomes = 52

$\therefore\ P(E) = \frac{4}{52} = \frac{1}{13}$

2. Here, the events E and F are complements of each other.

$\therefore P(E) + P(F) = 1$

$P(F) = 1 - \frac{1}{13}$

$= \frac{12}{13}$

Example 2: If the probability of an event A is 0.12 and B is 0.88 and they belong to the same set of observations, then show that A and B are complementary events.

Solution:

It is given that $P(A) = 0.12$ and $P(B) = 0.88$

Now, $P(A) + P(B) = 0.12 + 0.88 = 1$

∴ The events A and B are complementary events.

Example 3: Savita and Babita are playing badminton. The probability of Savita winning the match is 0.52. What is the probability of Babita winning the match?

Solution:

Let E be the event 'Savita winning the match' and F be the event 'Babita wining the match'.

It is given that $P(E) = 0.52$

Here, E and F are complementary events because if Babita wins the match, Savita will surely lose the match and vice versa.

∴ $P(E) + P(F) = 1$

$0.52 + P(F) = 1$

$P(F) = 1 - 0.52 = 0.48$

Thus, the probability of Babita winning the match is 0.48.

Example 4: In a box, there are 2 red, 5 blue, and 7 black marbles. One marble is drawn from the box at random. What is the probability that the marble drawn will be (i) red (ii) blue (iii) black (iv) not blue?

Solution:

Since the marble is drawn at random, all the marbles are equally likely to be drawn.

Total number of marbles = 2 + 5 + 7 = 14

Let A be the event 'the marble is red', B be the event 'the marble is blue' and C be the event 'the marble is black.

(i) Number of outcomes favourable to event A = 2

∴ $P(A) = \frac{2}{14} = \frac{1}{7}$

(ii) Number of outcomes favourable to event B = 5

$\therefore P(B) = \frac{5}{14}$

(iii) Number of outcomes favourable to event C = 7

$\therefore P(C) = \frac{7}{14} = \frac{1}{2}$

(iv) We have, $P(B) = \frac{5}{14}$

The event of drawing a marble which is not blue is the complement of event B.

$\therefore P(\overline{B}) = 1 - P(B) = 1 - \frac{5}{14} = \frac{9}{14}$

Thus, the probability of drawing a marble which is not blue is $\frac{9}{14}$.