Principle of Mathematical Induction

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The principle of mathematical induction can be stated as
Suppose there is a given statement P(*n*) involving the natural number *n* such that
(i) The statement is true for *n* = 1, i.e., P(1) is true, and
(ii) If the statement is true for *n* = *k* (where *k* is some positive integer), then the statement is true for *n* = *k* + 1, i.e., truth of P(*k*) implies the truth of P(*k* + 1).
Then, P(*n*) is true for all natural numbers *n*.

- Here, our assumption that the statement is true for $n = k$ is called **inductive hypothesis**.

Example 1: Prove that $9^{n+1} - 8n - 9$ is divisible by 8.

Solution:

Let the given statement be P(n), i.e.,

P(n) : $9^{n+1} - 8n - 9$ is divisible by 8

For $n = 1$, P(1): $9^{1+1} - 8(1) - 9 = 81 - 8 - 9 = 64$, which is divisible by 8

Thus, P(n) is true for $n = 1$.

Let P(n) be true for $n = k$, i.e., $9^{k+1} - 8k - 9$ is divisible by 8 for some natural number k.

Let $9^{k+1} - 8k - 9 = 8m$, where m is a natural number.

Now, we have to prove that P(k + 1) is true whenever P(k) is true.

$P(k + 1) = 9^{(k+1)+1} - 8(k + 1) - 9$

$= 9^{(k+2)} - 8k - 8 - 9$

$= 9^{(k+1)}.9 - 8k - 8 - 9$

$= 9^{(k+1)}(8 + 1) - 8k - 8 - 9$

$= 9^{(k+1)}(8 + 1) - 8k - 8 - 9$

$= 8.\ 9^{(k+1)} + 9^{(k+1)} - 8k - 8 - 9$

$= \{9^{(k+1)} - 8k - 9\} + 8\,(9^{(k+1)} - 1)$

$= 8m + 8\,(9^{(k+1)} - 1)$

$= 8\{m + (9^{(k+1)} - 1)\}$

Thus, P(k + 1) is divisible by 8.

Thus, by the principle of mathematical induction, the given statement is true for every positive integer n.

Example 2: Prove the following by the principle of mathematical induction.

$2^n > n^2$, where n is a positive integer such that $n > 4$.

Solution:

Let the given statement be P(n), i.e.,

P(n) : $2^n > n^2$ where $n > 4$

For $n = 5$,

$2^5 = 32$ and $5^2 = 25$

$\therefore 2^5 > 5^2$

Thus, P(n) is true for $n = 5$.

Let P(n) be true for $n = k$, i.e.,

$2^k > k^2$ … (1)

Now, we have to prove that P(k + 1) is true whenever P(k) is true, i.e. we have to prove that $2^{k+1} > (k+1)^2$.

From equation (1), we get

$2^k > k^2$

Multiplying both sides with 2, we obtain

$2 \times 2^k > 2 \times k^2$

$2^{k+1} > 2k^2$

$\therefore$To prove $2^{k+1} > (k+1)^2$, we only need to prove that $2k^2 > (k+1)^2$.

Let us assume $2k^2 > (k+1)^2$.

$\Rightarrow 2k^2 > k^2 + 2k + 1$

$\Rightarrow k^2 > 2k + 1$

$\Rightarrow k^2 - 2^k - 1 > 0$

$\Rightarrow (k-1)^2 - 2 > 0$

$\Rightarrow (k-1)^2 > 2$, which is true as $k > 4$

Hence, our assumption $2k^2 > (k+1)^2$ is correct and we have $2^{k+1} > (k+1)^2$.

Thus, P(n) is true for $n = k + 1$.

Thus, by the principle of mathematical induction, the given mathematical statement is true for every positive integer n.

Example 3: Prove the following by the principle of mathematical induction for every positive integer n.

2.5 + 4.7 + 6.9 + … + 2n(2n + 3) = $\frac{1}{3}n(n+1)(4n+11)$

Solution:

Let the given statement be P(n), i.e.,

P(n): 2.5 + 4.7 + 6.9 + … + 2n(2n + 3) = $\frac{1}{3}n(n+1)(4n+11)$

For $n = 1$, 2.5 = $\frac{1}{3}(1+1)(4+11) = \frac{1}{3}2.15 = 2.5$,which is true.

Let P(n) be true for $n = k$, i.e.,

2.5 + 4.7 + 6.9 + … + 2k(2k + 3) = $\frac{1}{3}k(k+1)(4k+11)$ … (1)

We now have to prove that P(k + 1) is true whenever P(k) is true.

2.5 + 4.7 + 6.9 + … + 2k(2k + 3) + 2(k + 1){2(k + 1) + 3}

={ 2.5 + 4.7 + 6.9 + … + 2k(2k + 3)} + 2(k + 1){2(k + 1) + 3}

= $\frac{1}{3}k(k+1)(4k+11)$ + 2(k + 1){2(k + 1) + 3} {from equation (1)}

$$=(k+1)\left\{\frac{1}{3}k(4k+11)+2(2k+5)\right\}$$

$$=\frac{1}{3}(k+1)\{4k^2+23k+30\}$$

$$=\frac{1}{3}(k+1)\{(k+2)(4k+15)\}$$

$$=\frac{1}{3}(k+1)\left[\{(k+1)+1\}\{4(k+1)+11\}\right]$$

Thus, P(n) is true for $n = k + 1$.

Thus, by the principle of mathematical induction, the given statement is true for every positive integer n.