

Binomial Theorem

Pascal's Triangle

- Some common expansions are given as
- $(a + b)^0 = 1$
- $(a + b)^1 = a + b$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a + b)^4 = (a + b)^2 (a + b)^2 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- The index of each expansion and the coefficients of the terms in the expansions are different. They however, share a relationship, which is given by **Pascal's Triangle**, which is shown below.

Index	Coefficients					
0	0C_0 (= 1)					
1		1C_0 (= 1)		1C_1 (= 1)		
2		2C_0 (= 1)	2C_1 (= 2)		2C_2 (= 1)	
3		3C_0 (= 1)	3C_1 (= 3)	3C_2 (= 3)		3C_3 (= 1)
4		4C_0 (= 1)	4C_1 (= 4)	4C_2 (= 6)	4C_3 (= 4)	4C_4 (= 1)
5	5C_0 (= 1)	5C_1 (= 5)	5C_2 (= 10)	5C_3 (= 10)	5C_4 (= 5)	5C_5 (= 1)

Pascal's triangle can be continued endlessly and can be used for writing the coefficients of the terms occurring in the expansion of $(a + b)^n$.

- For example, look at the row corresponding to index 5. It can be used for expanding $(a + b)^5$ as $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Binomial Theorem

- Binomial theorem is used for expanding the expressions of the type $(a + b)^n$, where n can be a very large positive integer.
- The binomial theorem states that the expansion of a binomial for any positive integer n is given by $(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \dots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$

$$\sum_{k=0}^n {}^nC_k a^{n-k} b^k$$

- The binomial theorem can also be stated as $(a + b)^n = \sum_{k=0}^n {}^nC_k a^{n-k} b^k$
- The coefficients nC_r occurring in the binomial theorem are known as binomial coefficients.
- There are $(n + 1)$ terms in the expansion of $(a + b)^n$.
- In the successive terms of the expansion, the index of a goes on decreasing by unity starting from n , whereas the index of b goes on increasing by unity starting from 0.
- In the expansion of $(a + b)^n$, the sum of indices of a and b in every term is n .
- Special cases of expansion can be obtained by taking different values of a and b .
 - Taking $a = x$ and $b = -y$:
 $(x - y)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2}y^2 - \dots + (-1)^n {}^nC_n y^n$
 - Taking $a = 1$ and $b = x$:
 $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$
 - Taking $a = 1$ and $b = 1$:
 $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_{n-1} + {}^nC_n$
 - Taking $a = 1$ and $b = -x$:
 $(1 - x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - \dots + (-1)^n {}^nC_n x^n$
 - Taking $a = 1$ and $b = -1$:
 $0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - \dots + (-1)^n {}^nC_n$

Solved Examples

Example 1: Write the expansion of the expression $\left(1 - \frac{3}{x}\right)^8$, where $x \neq 0$.

Solution:

Using Binomial theorem, we have

$$\begin{aligned} \left(1 - \frac{3}{x}\right)^8 &= {}^8C_0(1)^8 - {}^8C_1(1)^7\left(\frac{3}{x}\right) + {}^8C_2(1)^6\left(\frac{3}{x}\right)^2 - {}^8C_3(1)^5\left(\frac{3}{x}\right)^3 + {}^8C_4(1)^4\left(\frac{3}{x}\right)^4 \\ &\quad - {}^8C_5(1)^3\left(\frac{3}{x}\right)^5 + {}^8C_6(1)^2\left(\frac{3}{x}\right)^6 - {}^8C_7(1)\left(\frac{3}{x}\right)^7 + {}^8C_8\left(\frac{3}{x}\right)^8 \\ &= 1 - 8\left(\frac{3}{x}\right) + 28\left(\frac{3}{x}\right)^2 - 56\left(\frac{3}{x}\right)^3 + 70\left(\frac{3}{x}\right)^4 - 56\left(\frac{3}{x}\right)^5 + 28\left(\frac{3}{x}\right)^6 - 8\left(\frac{3}{x}\right)^7 + \left(\frac{3}{x}\right)^8 \end{aligned}$$

Example 2: Find the value of $(202)^4$.

Solution:

We can write 202 as $200 + 2$.

$$\therefore (202)^4 = (200 + 2)^4$$

On applying binomial theorem, we obtain

$$\begin{aligned} (202)^4 &= (200 + 2)^4 \\ &= {}^4C_0(200)^4 + {}^4C_1(200)^3(2) + {}^4C_2(200)^2(2)^2 + {}^4C_3(200)(2)^3 + {}^4C_4(2)^4 \\ &= (200)^4 + 4(200)^3(2) + 6(200)^2(2)^2 + 4(200)(2)^3 + (2)^4 \\ &= 160000000 + 64000000 + 960000 + 6400 + 16 \\ &= 1664966416 \end{aligned}$$

Example 3: Evaluate: $\left(1 + \frac{x}{2}\right)^5 + \left(1 - \frac{x}{2}\right)^5$.

Solution:

On using binomial theorem, we obtain

$$\left(1 + \frac{x}{2}\right)^5 = {}^5C_0(1)^5 + {}^5C_1(1)^4\left(\frac{x}{2}\right) + {}^5C_2(1)^3\left(\frac{x}{2}\right)^2 + {}^5C_3(1)^2\left(\frac{x}{2}\right)^3 + {}^5C_4(1)\left(\frac{x}{2}\right)^4 + {}^5C_5\left(\frac{x}{2}\right)^5$$

$$\left(1 - \frac{x}{2}\right)^5 = {}^5C_0(1)^5 - {}^5C_1(1)^4\left(\frac{x}{2}\right) + {}^5C_2(1)^3\left(\frac{x}{2}\right)^2 - {}^5C_3(1)^2\left(\frac{x}{2}\right)^3 + {}^5C_4(1)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$$

Thus,

$$\begin{aligned} & \left(1 + \frac{x}{2}\right)^5 + \left(1 - \frac{x}{2}\right)^5 \\ &= {}^5C_0(1)^5 + {}^5C_1(1)^4\left(\frac{x}{2}\right) + {}^5C_2(1)^3\left(\frac{x}{2}\right)^2 + {}^5C_3(1)^2\left(\frac{x}{2}\right)^3 + {}^5C_4(1)\left(\frac{x}{2}\right)^4 + {}^5C_5\left(\frac{x}{2}\right)^5 \\ & \quad + {}^5C_0(1)^5 - {}^5C_1(1)^4\left(\frac{x}{2}\right) + {}^5C_2(1)^3\left(\frac{x}{2}\right)^2 - {}^5C_3(1)^2\left(\frac{x}{2}\right)^3 + {}^5C_4(1)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5 \\ &= 2 \left[{}^5C_0(1)^5 + {}^5C_2(1)^3\left(\frac{x}{2}\right)^2 + {}^5C_4(1)\left(\frac{x}{2}\right)^4 \right] \\ &= 2 \left[1 + 10\left(\frac{x}{2}\right)^2 + 5\left(\frac{x}{2}\right)^4 \right] \\ &= 2 + 5x^2 + \frac{5}{8}x^4 \end{aligned}$$

General and Middle Term of A Binomial Expansion

- The $(r + 1)^{\text{th}}$ term or the **general term** of a binomial expansion is given by $T_{r+1} = {}^nC_r a^{n-r}b^r$
- For example: The 15th term in the expansion of $(5a + 3)^{25}$ is given by $T_{14+1} = {}^{25}C_{14} a^{25-14}b^{14} = {}^{25}C_{14} a^{11}b^{14}$
- To find the **middle term** of the expansion of $(a + b)^n$, the following formula is used:
- If n is even, then the number of terms in the expansion will be $n + 1$. Since n is even, then $(n + 1)$ is odd. Therefore, the middle term is $\left(\frac{n+1+1}{2}\right)^{\text{th}}$, i.e., the $\left(\frac{n}{2}+1\right)^{\text{th}}$ term.
- If n is odd, then $n + 1$ is even. Hence, there will be two middle terms in the expansion, namely the $\left(\frac{n+1}{2}\right)^{\text{th}}$ term and the $\left(\frac{n+1}{2}+1\right)^{\text{th}}$ term.

- In the expansion of $\left(x + \frac{1}{x}\right)^{2n}$, where $x \neq 0$, the middle term is $\left(\frac{2n+1+1}{2}\right)^{\text{th}}$, i.e., the $(n + 1)^{\text{th}}$ term, as $2n$ is even.

Example 1: Find the term independent of p in the expansion of $\left(2p - \frac{1}{p}\right)^{16}$.

Solution:

We know that the general term i.e., the $(r + 1)^{\text{th}}$ term of the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Hence,

$$T_{r+1} = {}^{16}C_r (2p)^{16-r} \left(-\frac{1}{p}\right)^r = (-1)^r {}^{16}C_r (2)^{16-r} (p)^{16-r} \left(\frac{1}{p}\right)^r = (-1)^r {}^{16}C_r (2)^{16-r} (p)^{16-2r}$$

The term will be independent of p , if the index of p is zero i.e., $16 - 2r = 0$.

This gives $r = 8$.

Hence, the 9th term is independent of p and it is given

$$\text{by } (-1)^8 {}^{16}C_8 (2)^{16-8} (p)^{16-2 \times 8} = \frac{16!}{8!8!} (2)^8 (p)^0 = 12870 \times (2)^8$$

Example 2: In the expansion of $(2p + n)^7$, where n is an integer, the third and fourth terms are $6048 p^5$ and $15120 p^4$ respectively. Find the value of n .

Solution: We know that the general term i.e., the $(r + 1)^{\text{th}}$ term of the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Thus,

$$\begin{aligned} \text{Third term} &= T_{2+1} \\ &= {}^7C_2 (2p)^{7-2} n^2 \\ &= 21 \times (2)^5 p^5 n^2 \end{aligned}$$

The third term is given as $6048 p^5$. Therefore,

$$21 \times (2)^5 p^5 n^2 = 6048 p^5$$

$$\Rightarrow n^2 = 9 \dots (1)$$

$$\begin{aligned} \text{Fourth term, } T_{3+1} &= {}^7C_3(2p)^{7-3}n^3 \\ &= 35 \times (2)^4 p^4 n^3 \end{aligned}$$

The fourth term is given as $15120 p^4$. Therefore,

$$35 \times (2)^4 p^4 n^3 = 15120 p^4$$

$$\Rightarrow n^3 = 27 \dots (2)$$

On dividing equation (2) by equation (1), we obtain

$$n = \frac{27}{9} = 3$$

Thus, the value of n is 3.

Example 3: Find the coefficient of x^2 in the expansion of $\left(\frac{1}{2} - \sqrt{x}\right)^{10}$.

Solution: Suppose x^2 occurs in the $(r + 1)^{\text{th}}$ term of the expansion of $\left(\frac{1}{2} - \sqrt{x}\right)^{10}$.

Now, $T_{r+1} = {}^nC_r a^{n-r} b^r$

$$\therefore T_{r+1} = {}^{10}C_r \left(\frac{1}{2}\right)^{10-r} (-\sqrt{x})^r = {}^{10}C_r \left(\frac{1}{2}\right)^{10-r} (-1)^r (\sqrt{x})^r$$

Comparing the indices of x in x^2 and T_{r+1} , we obtain $r = 4$.

Thus, the coefficient of x^2 in the expansion of $\left(\frac{1}{2} - \sqrt{x}\right)^{10}$ is given by

$$T_{4+1} = {}^{10}C_4 \left(\frac{1}{2}\right)^{10-4} (-1)^4 = {}^{10}C_4 \left(\frac{1}{2}\right)^6 = \frac{105}{32}$$