Mathematical Reasoning

Statement

- In any language, a '**sentence**' is a group of words that are put together to mean something. It is the basic unit of a language and expresses complete thought.

- The expression "Neha is a good girl" expresses a complete thought. So, it is a sentence. But the expression 'Sun' does not express a complete thought. So, it is not a sentence.

- A sentence is called a mathematically acceptable **statement** if it is either **true** or **false**,but **not both**.

Example 1: Which of the following sentences are statements? Give reasons for your answers.

1. If the terms *a*, *b* and *c* are in both A.P. and G.P., then *a* = *b* = *c*.
2. $5-2\sqrt{11}$ is a rational number.
3. There are 30 days in the month next to the current month.
4. Oil floats on water.
5. What is the sum of two odd numbers?

Solution:

1. If the terms *a*, *b* and *c* are in A.P., then
   $b=\dfrac{a+c}{2}$
   If the terms *a*, *b* and *c* are in G.P., then
   $b^2=ac$
   $\Rightarrow \left(\dfrac{a+c}{2}\right)^2=ac$
   $\Rightarrow (a+c)^2-4ac=0$
   $\Rightarrow (a-c)^2=0$
   $\Rightarrow a-c=0$
   $\Rightarrow a=c$

$\therefore b = \frac{c+c}{2} = c$

This shows that then $a = b = c$. Since the given sentence is true in all cases, it is a statement.

2. $5-2\sqrt{11}$ is an irrational number, i.e., not a rational number. Hence, the given sentence is a statement.

3. In the given sentence, there is no information as to which month is being referred to as the current month. So, it is not possible to say with surety whether the month next to this month has 30 days or not. Hence, it is not a statement.

4. The sentence, "Oil floats on water" is scientifically true. It is true in all cases, so it is a statement.

5. This sentence is a question. Hence, it is not a statement.

Negation of a Statement

- The denial of a statement is called the **negation** of that statement.
- The negation of a statement p is denoted as $\sim p$.
- Phrases like "It is not the case" or "It is false that" are also used while forming the negation of a statement.
- If we have a statement p: "There are 31 days in July", then its negation can be written as $\sim p$: "There are not 31 days in July". The other ways of writing $\sim p$ are "It is false that there are 31 days in July" or "It is not the case that there are 31 days in July".
- If we have a statement q: "Every leaf in this plant is green in colour", then we can write the negation of q as $\sim q$: "Every leaf in this plant is not green in colour".

Example 1: Write the negation of the following statements.

1. a being a prime number, a^2 is a composite number.

2. Every number of the form $a + ib$ is a complex number, where a and b are real numbers and $b \neq 0$.

3. Rourkela is an industrial area in Orissa.

Solution:

1. The negation of the statement "a being a prime number, a^2 is a composite number" is "a being a prime number, a^2 is not a composite number" or "a being a prime number, a^2 is a prime number" or "It is false that a being a prime number, a^2 is a composite number" or "It is not the case that a being a prime number, a^2 is a composite number".

2. The negation of the statement "Every number of the form $a + ib$ is a complex number, where a and b are real numbers and $b \neq 0$" is "It is not the case that every number of the form $a + ib$ is a complex number, where a and b are real numbers and $b \neq 0$".

3. The negation of the statement "Rourkela is an industrial area in Orissa" is "Rourkela is not an industrial area in Orissa" or "It is false that Rourkela is an industrial area in Orissa".

Compound Sentence and its Components

- A sentence made up of two or more statements is called a **compound sentence (statement)**.

- The statement "$6! = 540$ or $3^5 = 243$" is a compound statement since it is made up of two statements "$6! = 540$" and "$3^5 = 243$".

- Some of the terms associated with a compound statement are components, and connectors or connecting words.

By looking at the above video, we have learnt that

- Each statement of the compound statement is called its **component**.

- The components are joined by **connectors** or **connecting words**.

- The connectors or the connecting words of a compound statement are "And" and "Or".

- For example, for the compound statement "$6! = 540$ or $3^5 = 243$", the components are "$6! = 540$" and "$3^5 = 243$". Here, the connectors or the connecting word for the given compound statement is "Or".

Example 1: In the following statements, identify the connecting words. Also, break them into their components.

1. 504 is a common multiple of exponents of 2 or 3.

2. Three lines have no common point or three common points.

3. Pranav may take a burger and a cup of lemon tea.

Solution:

1. The connecting word of the given statement is ***or***.
   The components *p* and *q* of the given statement are as follows:
   p: 504 is a common multiple of exponents of 2.
   q: 504 is a common multiple of exponents of 3.

2. The connecting word of the given statement is ***or***.
   The components *p* and *q* of the given statement are as follows:
   p: Three lines have no common point.
   q: Three lines have three common points.

3. The connecting word of the given statement is ***and***.
   The components *p* and *q* of the given statement are as follows:
   p: Pranav may take a burger.
   q: Pranav may take a cup of lemon tea.

Example 2: Make a compound statement by using its components *r* and *s*, where *r* and *s* are given by

r: The zeroes of the polynomial $x^3 - 6x^2 + 11x - 6$ are 1, 2 and 3

s: For $x < y$, $\frac{1}{x} > \frac{1}{y}$. Use the connectors ***or*** and ***and***.

Solution:

The components *r* and *s* of the compound statement are

r: The zeroes of the polynomial $x^3 - 6x^2 + 11x - 6$ are 1, 2 and 3

s: For $x < y$, $\frac{1}{x} > \frac{1}{y}$

Now,

r **and** *s*: The zeroes of the polynomial $x^3 - 6x^2 + 11x - 6$ are 1, 2 and 3 and for $x < y$, $\frac{1}{x} > \frac{1}{y}$.

r **or** *s*: The zeroes of the polynomial $x^3 - 6x^2 + 11x - 6$ are 1, 2 and 3 or for $x < y$, $\frac{1}{x} > \frac{1}{y}$.

Example 3: If p and q are the components of a compound statement, find $\sim p$ and q, given that

$\sim q$ or p: For each $a \in$ R, $a + 0 = a$ or 45 is not a multiple of 10.

Solution:

It is given that

$\sim q$ or p: For each $a \in$ R, $a + 0 = a$ or 45 is not a multiple of 10.

$\Rightarrow \sim q$: For each $a \in$ R, $a + 0 = a$.

p: 45 is not a multiple of 10.

$\Rightarrow q$: There exists at least one $a \in$ R, such that $a + 0 \neq a$

$\sim p$: 45 is a multiple of 10.

$\sim p$ and q: 45 is a multiple of 10 and there exists at least one $a \in$ R, such that $a + 0 \neq a$.

Role of Connectives "And" and "Or"

Validating Compound Statements with the Connectives "And" and "Or"

- The compound statement with the connective "and" is true if all of its component statements are true, else it is false.

- For the compound statement, x: "$\sqrt{3}+\sqrt{5}$ is a rational number and 36 has 9 factors", the connective is "and" and its components p and q are
  p: $\sqrt{3}+\sqrt{5}$ is a rational number.
  q: 36 has 9 factors.

  Here, you may observe that the statement p is false since $\sqrt{3}+\sqrt{5}$ is an irrational number, and the statement q is true since 36 does have 9 factors. They are 1, 2, 3, 4, 6, 9, 12, 18 and 36. Since both components p and q of the statement x are not true, x is false.

- The compound statement with the connective "or" is false if all of its component statements are false, else it is true.

- For the compound statement, y: "HCF (a, b) × LCM $(a, b) = ab$ or the root of the equation $x+\frac{1}{x}=2$ is 1", the connective is "or" and its components r and s are
  r: HCF (a, b) × LCM $(a, b) = ab$.

s: The root of the equation $x+\frac{1}{x}=2$ is 1.
Here, you may observe that both components of *y* are true. Hence, *y* is true.

"Exclusive Or" and "Inclusive Or"

- An exclusive "Or" is used in a compound statement with the connective "Or" if exactly one of its components is true, or to make the statement meaningful, only one component (not both) is taken into consideration.

- The statement "You may take a pizza or a burger" implies that if a person takes a pizza, then he/she will not take a burger, or if he/she will take a burger, then he/she will not take a pizza, i.e., there is only one choice in this statement, or one component in this statement is true, but not both. In this statement, we can say that an excusive "Or" is used.

- An inclusive "Or" is used in a compound sentence, if all its components are true, or to make the statement meaningful, all the components are taken into consideration.

- The statement "The sum of all interior angles or exterior angles of a quadrilateral is 360°" shows that the sum of all interior angles of a quadrilateral is 360° and so is the case with the exterior angles of a quadrilateral, i.e., both the components of the statement are true. In this statement, we can say that an inclusive "Or" is used.

Solved Examples

Example 1: Check whether the following compound statements are true or false.

1. The maximum value of $2\sin x+\sqrt{5}\cos x$ is 3 and $\lim_{x\to\frac{3}{2}}\frac{128x^7-2187}{8x^3-27}=189$.

2. Prime numbers are the positive odd integers or ratio of two rational numbers is a rational number.

3. $\sqrt{a+ib}=\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}+i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$ and cos 0° × cos 1° × cos 2°…. cos 180° = 1.

Solution:

1. The components *p* and *q* of the given compound statement are
   p: The maximum value of $2\sin x+\sqrt{5}\cos x$ is 3.

q: $\lim_{x \to \frac{3}{2}} \frac{128x^7 - 2187}{8x^3 - 27} = \frac{189}{16}$

The components are connected by the connective **And**. The given statement will be true if both the components are true. Let us check this.

<u>For *p*</u>

$\sqrt{(2)^2 + (\sqrt{5})^2} = 3$

Let 2 = 3 cos *y* and $\sqrt{5}$ = 3 sin *y*.

$$\therefore 2\sin x + \sqrt{5}\cos x = 3\cos y \sin x + 3\sin y \cos x$$
$$= 3\sin(x + y)$$

The maximum value of sin (*x* + *y*) is 1.

So, the maximum value of $2\sin x + \sqrt{5}\cos x$ is 3 × 1 = 3.

Hence, *p* is true.

<u>For *q*</u>

$$\lim_{x \to \frac{3}{2}} \frac{128x^7 - 2187}{8x^3 - 27} = \lim_{x \to \frac{3}{2}} \frac{128\left[(x)^7 - \left(\frac{3}{2}\right)^7\right]}{8\left[(x)^3 - \left(\frac{3}{2}\right)^3\right]}$$

$$= 16 \frac{\lim_{x \to \frac{3}{2}} \frac{(x)^7 - \left(\frac{3}{2}\right)^7}{x - \frac{3}{2}}}{\lim_{x \to \frac{3}{2}} \frac{(x)^3 - \left(\frac{3}{2}\right)^3}{x - \frac{3}{2}}}$$

$$= 16 \times \frac{7 \times \left(\frac{3}{2}\right)^6}{3 \times \left(\frac{3}{2}\right)^2}$$

$$= 189$$

Hence, *p* is also true.

Therefore, the given statement is true.

2. The components *r* and *s* of the given statement are

r: Prime numbers are positive odd integers.

s: Ratio of two rational numbers is a rational number.

The components are connected by the connective **Or**. The given statement will be true if either of the components is true, or both of them are true. Let us check this.

For *r*

We know that 2 is a prime number and that it is also an even number. This shows that *r* is false.

For *s*

$\frac{-2}{7}$ and 0 are rational numbers.

Now, $\frac{\frac{-2}{7}}{0} = \infty$. This cannot be defined. So, ratio of two rational numbers is not always a rational number. This shows that *s* is false.

Since, both *r* and *s* are false, the given statement is also false.

3. The components *u* and *v* of the given statement are

u: $\sqrt{a+ib} = \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$

v: cos 0° × cos 1° × cos 2°.... cos 180° = 1.

The components are connected by the connective **And**. The given statement will be true if both the components are true. Let us check this.

For *u*

$\sqrt{a+ib} = \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$ if $a+ib = \left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right)^2$

Now,

$$\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right)^2 = \left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}\right)^2 + \left(i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right)^2$$

$$+2\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}\right)\left(i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right)$$

$$= \frac{\sqrt{a^2+b^2}+a}{2} - \frac{\sqrt{a^2+b^2}-a}{2} + 2i\sqrt{\frac{b^2}{4}}$$

$$= a+ib$$

So, *u* is true.

For *v*

We know cos 90° = 0.

Now, cos 0° × cos 1° × cos 2°.... cos 180°

= cos 0° × cos 1° × cos 2°.... cos 90°.... cos 180°= 1 × cos 1° × cos 2°.... 0.... (−1)

= 0 ≠ 1

So, *v* is also false.

Hence, the given statement is false.

Example 2: State whether the **Or** used in each of the following statements is “**Exclusive”** or “**Inclusive”**.

1. A triangle ABC with ∠A = 45° and ∠B = 45° is an isosceles triangle or a right angle triangle.

2. The numbers divisible by 30 are divisible by 6 or 15.

Solution:

1. In ΔABC, ∠A = 45° and ∠B = 45°, so, ∠C = 180° − (∠A + ∠B) = 180° − 90° = 90°. Clearly, ΔABC is isosceles and right-angled. Hence, the ‘**Or’** used in this statement is ‘**Inclusive’**.

2. 6 and 15 are the factors of 30. Clearly, if a number is divisible by 30, then it is divisible by both 6 as well as 15. Hence, the ‘**Or’** used in this statement is ‘**Inclusive’**.

Role of Quantifiers in a Compound Statement

Quantifiers

- There are some compound statements that contain phrases like “There exists”, “For every”, “For all”. These phrases are called **quantifiers**.

- The quantifier for the statement “For every prime number x, $n \in N$, x^n has exactly (n + 1) factors” is “For every”. This statement is equivalent to “If A is the set of all prime numbers, then for any number x in the set A and $n \in N$, x^n has exactly (n + 1) factors”.

- The quantifier for the statement “There exists a number which is divisible by 4 and 7, but not by 2” is “There exists”. This statement is equivalent to "Out of all numbers that are divisible by both 4 and 7, there is at least one number which is not divisible by 2”.

Validating a Statement with a Quantifier

- The validity of a statement with a quantifier is calculated just by analysing the statement logically and mathematically.

- To find the validity of the statement, p: “For every prime number x, $n \in N$, x^n has exactly (n + 1) factors”, we may proceed as follows:
  Let x be a prime number. For $n \in N$, the factors of x^n are x^0 (= 1), x^1, x^2 …. x^n. Clearly, x^n has exactly (n + 1) factors. Hence, the statement p is true.

- To find the validity of the statement, q: “There exists a number which is divisible by 4 and 7, but not by 2”, we may proceed as follows:
  If a number is divisible by 4 and 7, then that number is divisible by 28 (L.C.M. of 4 and 7).

Since 2 is a factor of 28. So, the number must be divisible by 2. This shows that statement q is false.

Solved Examples

Example 1: Identify the quantifiers in the following statements, and check the validity of these statements.

1. There exists a solution to the equation tan x + cot x = $\sqrt{5}$.

2. For every $x, y \in$ R, $(x+iy)(x-iy)$ is a rational number.

3. For all $n \in$ N and $n \geq 2$, $4n^4+1$ is a composite number.

4. There exists an even positive integer which is a multiple of 5, but not 4.

Solution:

1. The quantifier in the given statement is "There exists".

   tan x + cot x = $\sqrt{5}$

   $\Rightarrow \frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=\sqrt{5}$

   $\Rightarrow \frac{\sin^2 x+\cos^2 x}{\sin x\cos x}=\sqrt{5}$

   $\Rightarrow \frac{1}{\sin x\cos x}=\sqrt{5}$

   $\Rightarrow 2\sin x\cos x=\sin 2x=\frac{2}{\sqrt{5}}$

   (By multiplying both sides by 2)

   But $\sqrt{5}>2$; so, $\sin 2x<1$

   Hence, there exists a solution to the equation tan x + cot x = $\sqrt{5}$. This shows that the given statement is true.

2. The quantifier in the given statement is "For every".

   Let $x=\sqrt{\sqrt{2}+\sqrt{3}}$ and $y=\sqrt{\sqrt{5}-\sqrt{3}}$

$$\text{Now, } (x+iy)(x-iy) = x^2 + y^2 = \left(\sqrt{2}+\sqrt{3}\right)^2 + \left(\sqrt{5}-\sqrt{3}\right)^2$$
$$= \left(5+2\sqrt{6}\right) + \left(8-2\sqrt{15}\right)$$
$$= 13 + 2\sqrt{3}(\sqrt{2}-\sqrt{5})$$

This is not a rational number. Therefore, the given statement is false.

3. The quantifier in the given statement is "For all".

$$4n^4 + 1 = \left(2n^2\right)^2 + 1$$
$$= \left(2n^2\right)^2 + (1)^2 + 2(2n^2)(1) - 2(2n^2)(1)$$
$$= \left(2n^2+1\right)^2 - 4n^2$$
$$= \left(2n^2+1\right)^2 - (2n)^2$$
$$= \left(2n^2+1+2n\right)\left(2n^2+1-2n\right)$$
$$= \left[(n+1)^2 + n^2\right]\left[(n-1)^2 + n^2\right]$$

For $n \geq 2$, both $(n+1)^2 + n^2$ and $(n-1)^2 + n^2$ are two natural numbers other than 1.
Since $4n^4 + 1$ is expressed as the product of two natural numbers other than 1, it is a composite number. Hence, the given statement is true.

4. The quantifier in the given statement is "There exists".
10 is an even number which is multiple of 5, but not 4.
Therefore, the given statement is true.

Implications

Implication of "if-then" and "only if"

- The statements with a phrase like "if-then" are called **statements of implications of "if-then"**.

- Such statements are widely used in Mathematics as well as in our day-to-day lives.

  For example, a statement like "If $\frac{a+b+c}{3} = b$, then a, b and c are in A.P" is used in Mathematics whereas a statement like "If I go to the market, then I will get vegetables" is used in our daily lives.

- Apart from this concept, we may write p implies q, i.e., $p \Rightarrow q$ in the following ways:

1. p is a sufficient condition for q.

2. q is a necessary condition for p.

3. p only if q. (So, the implications of "if-then" and "only if" are equal)

4. $\sim q \Rightarrow \sim p$ (Contrapositive)

- Mathematically, $p \Rightarrow q$ states that if p is true, then q is true; however, it states nothing about the truth or falsehood of p if q is true.

Implication of "if and only if"

- The statements with a phrase like "if and only if" are called **statements of implications of "if and only if"**.

- Symbolically, we write a statement "p if and only if q" as $p \Leftrightarrow q$.

- The statement "A number is a multiple of 6 if and only if it is a multiple of both 2 and 3" is a statement of implication of "if and only if". This is in the form of $p \Leftrightarrow q$, where
  p: A number is a multiple of 6.
  q: A number is a multiple of both 2 and 3.

- Mathematically, $p \Leftrightarrow q$ states that if "p is true, then q is true" and "if q is true, then p is true".

- $p \Leftrightarrow q$ is equivalent to saying "$p \Rightarrow q$ and $q \Rightarrow p$".

- The statement "A number is a multiple of 6 if and only if it is a multiple of both 2 and 3" is broken into two statements of implication of "if-then" as "If a number is a multiple of 6, then it is a multiple of both 2 and 3" and "If a number is a multiple of both 2 and 3, then it is a multiple of 6".

- The other ways of writing a statement $p \Leftrightarrow q$ are

1. $q \Leftrightarrow p$

2. p is a necessary and sufficient condition for q, and vice versa.

Solved Examples

Example 1: Given below are three pairs of statements. Combine each pair using "if-then". Write their converse and contrapositive.

1. p: a, b, c and d are in proportion.
   q: $ad = bc$

2. p: You water the plants regularly.
   q: The plants will not die.

3. p: x and y are multiples of 4.
   q: $x + y$ is a multiple of 4.

Solution:

1. We have two statements p and q as
   p: a, b, c and d are in proportion.
   q: $ad = bc$
   The combination of the statements p and q using "if-then" ($p \Rightarrow q$) is as follows:
   ($p \Rightarrow q$): If a, b, c and d are in proportion, then $ad = bc$.
   The converse of this statement ($q \Rightarrow p$) is as follows:
   ($q \Rightarrow p$): If $ad = bc$, then a, b, c and d are in proportion.
   We have,
   $\sim p$: a, b, c and d are not in proportion.
   $\sim q$: $ad \neq bc$
   Hence, the contrapositive of the statement $p \Rightarrow q$ can be written as
   ($\sim q \Rightarrow \sim p$): If $ad \neq bc$, then a, b, c and d are not in proportion..

2. We have two statements p and q as
   p: You water the plants regularly.
   q: The plants will not die.
   The combination of the statements p and q using "if-then" ($p \Rightarrow q$) is as follows:
   ($p \Rightarrow q$): If you water the plants regularly, then they will not die.
   The converse of this statement ($q \Rightarrow p$) is as follows:
   ($q \Rightarrow p$): If the plants do not die, then you water them regularly.
   We have,
   $\sim p$: You do not water the plants regularly.
   $\sim q$: The plants will die.
   Hence, the contrapositive of the statement $p \Rightarrow q$ can be written as
   ($\sim q \Rightarrow \sim p$): If the plants die, then you do not water them regularly.

3. We have two statements p and q as
   p: x and y are multiples of 4.
   q: $x + y$ is a multiple of 4.
   The combination of the statements p and q using "if-then" ($p \Rightarrow q$) is as follows:
   ($p \Rightarrow q$): If x and y are multiples of 4, then $x + y$ is a multiple of 4.
   The converse of this statement ($q \Rightarrow p$) is as follows:
   ($q \Rightarrow p$): If $x + y$ is a multiple of 4, then x and y are multiples of 4.
   We have,
   $\sim p$: x and y are not multiples of 4.
   $\sim q$: $x + y$ is not a multiple of 4.

Hence, the contrapositive of the statement $p \Rightarrow q$ can be written as
($\sim q \Rightarrow \sim p$): If $x + y$ is not a multiple of 4, then x and y are not multiples of 4.

Example 2: Write the necessary and sufficient conditions for the following statement:

"If x is a perfect square, then x has total odd number of factors."

Solution:

The given statement is an implication of "if-then", i.e., in the form of $p \Rightarrow q$, where

p: x is a perfect square.

q: x has total odd number of factors.

We know that for a statement $p \Rightarrow q$,

1. p is a sufficient condition for q.
2. q is a necessary condition for p.

Hence, the statement

1. "x is a perfect square" is the sufficient condition for the statement "x has total odd number of factors".
2. "x has total odd number of factors" is the necessary condition for the statement "x is a perfect square".

Example 3: Break the following "if and only if" statement into a pair of "if-then" statements:

"p divides a^2 if and only if p divides a, where p is a prime number."

Solution:

The given "if and only if" statement can be broken into a pair of "if-then" statements as

1. If p divides a^2, then p divides a, where p is a prime number.
2. If p divides a, then p divides a^2, where p is a prime number.

Validity of "If-then" and "If and only if" Statements

Validating the implication of an "if-then" statement

- The validity of a statement of implication of "if-then" is checked by two methods. They are
  1. Direct Method

  2. Contrapositive Method

Validating the implication of an "if and only if" statement

- An implication statement of "if and only if" of the form $p \Leftrightarrow q$ is said to be true, if both $p \Rightarrow q$ and $q \Rightarrow p$ are true.

- We can check the validity of $p \Rightarrow q$ and $q \Rightarrow p$ by using either the direct method or the contrapositive method, whichever is desirable.

Example 1: Using both the "Direct Method" and the "Contrapositive Method", show that

1. If $x > y$, then $\frac{1}{x} < \frac{1}{y}$, where $x, y \in Z^+$.
2. If A ∩ B = Φ, then n (A ∩ B) = n(A) + n(B), where A and B are non-empty sets.

Solution:

1. For $x, y \in Z^+$, we have $xy \in Z^+$.
   $\Rightarrow \frac{1}{xy} \in Z^+$
   The given statement of implication of "if-then" is of the form $p \Rightarrow q$, where
   p: $x > y$, where $x, y \in Z^+$.
   q: $\frac{1}{x} < \frac{1}{y}$, where $x, y \in Z^+$.

   Direct Method

   In this method, we have to assume p to be true, and then we have to prove that q is true.

   For this, let us assume $x > y$, where $x, y \in Z^+$. Since $\frac{1}{xy} \in Z^+$

$\therefore \frac{x}{xy} > \frac{y}{xy}$

$\Rightarrow \frac{1}{y} > \frac{1}{x}$

$\Rightarrow \frac{1}{x} < \frac{1}{y}$

So, $p \Rightarrow q$ is true.

Contrapositive Method

In this method, we have to assume q to be false, and then we have to prove that p is false.

For this, let us assume $\frac{1}{x}$ is not lesser than $\frac{1}{y}$, i.e., $\frac{1}{x} \geq \frac{1}{y}$, where $x, y \in Z^+$.

Since $xy \in Z^+$,

$\frac{1}{x} \times xy \geq \frac{1}{y} \times xy$

$\Rightarrow y \geq x$

So, p is false.

Hence, $\sim q \Rightarrow \sim p$.

Therefore, $p \Rightarrow q$ is true.

2. The given statement of implication of "if-then" is of the form $p \Rightarrow q$, where

p: $A \cap B = \Phi$, where A and B are non-empty sets.

q: $n(A \cap B) = n(A) + n(B)$, where A and B are non-empty sets.

Direct Method

In this method, we have to assume p to be true, and then we have to prove that q is true.

For this, let us assume for two non-empty sets A and B, $A \cap B = \Phi$

$\therefore n(A \cap B) = 0$

We know that

n (A ∩ B) = n(A) + n(B) − n(A∩B)

⇒ n (A ∩ B) = n(A) + n(B) − 0 = n(A) + n(B)

Hence, $p \Rightarrow q$ is true.

Contrapositive Method

In this method, we have to assume q to be false, and then we have to prove that p is false.

For this, let us assume n (A ∩ B) ≠ n(A) + n(B)

But n (A ∩ B) = n(A) + n(B) − n(A∩B)

Hence, n (A ∩ B) = n(A) + n(B) − n(A∩B) ≠ n(A) + n(B)

⇒ n(A∩B) ≠ 0

⇒ A∩B ≠ Φ

So, p is false, i.e., $\sim q \Rightarrow \sim p$ is true.

Hence, $p \Rightarrow q$ is true.

Example 2: Show that the statement "A triangle with sides a, b and hypotenuse c is a right triangle, if and only if $a^2 + b^2 = c^2$" is true.

Solution:
The given statement is an implication of "if and only if" of the form $p \Leftrightarrow q$, where

p: A triangle with sides a, b and hypotenuse c is a right triangle.

q: A triangle with sides a, b and c is such that $a^2 + b^2 = c^2$.

For this, we have to prove both $p \Rightarrow q$ (If a triangle with sides a, b and hypotenuse c is a right triangle, then $a^2 + b^2 = c^2$) and $q \Rightarrow p$ ((If a triangle with sides a, b and c is such that $a^2 + b^2 = c^2$, then the triangle is a right triangle with hypotenuse c) are true.

Let us start with $p \Rightarrow q$. Let us assume p to be true.

This means that the triangle with sides a, b and hypotenuse c is a right triangle.

So, by using Pythagoras theorem for this triangle, we can say that

$a^2 + b^2 = c^2$.

So, q is true.

Hence, $p \Rightarrow q$ is true.

Now, we need to prove $q \Rightarrow p$. For this, let us assume q to be true.

This means we have a triangle with sides a, b and c, such that $a^2 + b^2 = c^2$.

So, by using converse of Pythagoras Theorem, we can say that

The given triangle is a right triangle with hypotenuse c

So, p is true.

Hence, $q \Rightarrow p$ is also true.

Since both $p \Rightarrow q$ and $q \Rightarrow p$ are true, the given statement $p \Leftrightarrow q$ is true.

Validation of a Statement by the Method of Contradiction and by Giving a Counter Example

Validation of a Statement by using the Method of Contradiction

- To prove a statement p to be true by the **Contradiction Method**, first of all, we assume that p is not true, i.e., $\sim p$ is true. Then, we arrive at some result which contradicts our assumption. On the basis of this contradiction, we can say that p is true.

Disproving a Statement by Giving a Counter Example

- To disprove a statement or to prove a statement false, we give the example of a situation where the statement is not valid. Such an example is called a **counter example**. This method is known as the method of disproving a statement by giving a counter example.

- The statement "The number of factors of a positive perfect cube integer other than 1 is a multiple of 4" can be disproved as follows:
  The above statement states that if x is a perfect cube, i.e., $x = a^3$, then there are $4n$ factors of x, where a and n are natural numbers.
  There are 7 factors of the perfect cube 64 ($= 4^3$). They are 1, 2, 4, 8, 16, 32 and 64. However, 7 is not a multiple of 4. So, the given statement does not hold true for the number 64.
  Hence, we can say that the given statement is false.

- Generating examples in favour of a statement does not provide validity to the statement (we have seen it in the case of the above statement). Counter examples are always used for disproving a statement.

- The statement "The numbers of factors of a positive perfect cube integer other than 1 is a multiple of 4" is true for the numbers 8 and 27. However, it is not true for 64. So, the given statement is false. Hence, we cannot say that the given statement is true.

Solved Examples

Example 1: Prove by the method of contradiction that "If *a* and *b* are two positive odd integers, such that $a > b$, then among the positive integers $\frac{a+b}{2}$ and $\frac{a-b}{2}$, one is odd and the other is even.

Solution:

Since *a* and *b* are two positive odd integers,

$a = 2x + 1$, where *x* is a natural number.

$b = 2y + 1$, where *y* is a whole number.

Let us assume that the given statement is false. For this, let us assume that "If *a* and *b* are two positive odd integers, such that $a > b$, then both $\frac{a+b}{2}$ and $\frac{a-b}{2}$ are odd or even positive integers".

$$\text{Now, } \frac{a+b}{2} = \frac{(2x+1)+(2y+1)}{2} = x+y+1 \text{ is a positive integer}$$
$$\text{And } \frac{a-b}{2} = \frac{(2x+1)-(2y+1)}{2} = x-y \text{ is a positive integer}$$

$$\therefore \frac{a+b}{2} - \frac{a-b}{2} = b \text{ is an odd positive integer}$$

However, if both $\frac{a+b}{2}$ and $\frac{a-b}{2}$ are odd or even positive integers, then $\frac{a+b}{2} - \frac{a-b}{2}$ is an even integer. This contradicts our assumption that "If *a* and *b* are two positive odd integers, such that $a > b$, then both $\frac{a+b}{2}$ and $\frac{a-b}{2}$ are odd or even positive integers".

So, our assumption was wrong. Hence, by the method of contradiction we can say that "If a and b are two positive odd integers, such that $a > b$, then among the positive integers $\frac{a+b}{2}$ and $\frac{a-b}{2}$, one is odd and the other is even.

Example 2: By giving a counter example, show that the following statement is not true.

p: $n^3 - n$ is divisible by 12, for $n \in N$, such that $n > 2$.

Solution:

For $n = 6$, we have

$n^3 - n = 210 = 6 \times 35$, which is not divisible by 12.

Therefore, the given statement is not true.