INVERSE TRIGONOMETRIC FUNCTIONS

BASIC CONCEPTS

INVERSE CIRCULAR FUNCTIONS

	Function	Domain	Range
1.	$y = \sin^{-1} x$ iff $x = \sin y$	$-1 \le x \le 1,$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$ iff $x = \cos y$	$-1 \le x \le 1$	$[0, \pi]$
3.	$y = \tan^{-1} x$ iff $x = \tan y$	$-\infty < x < \infty$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1} x$ iff $x = \cot y$	$-\infty < x < \infty$	$[0, \pi]$
5.	$y = \text{cosec}^{-1} x$ iff $x = \text{cosec } y$	$(-\infty, -1] \cup [1, \infty]$	$\left[-\frac{\pi}{2}.0\right) \cup \left(0, \frac{\pi}{2}\right]$
6.	$y = \sec^{-1} x$ iff $x = \sec y$	$(-\infty, -1] \cup [1, \infty]$	$\left[0.\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Note...

(i) $\text{Sin}^{-1} x$ & $\tan^{-1} x$ are increasing functions in their domain.

(ii) $\text{Cos}^{-1} x$ & $\cot^{-1} x$ are decreasing functions in over domain.

PROPERTY – I

(i) $\sin^{-1} x + \cos^{1} x = \pi/2$, for all $x \in [-1, 1]$

Sol. Let, $\sin^{-1} x = \theta$ ... (i)

then, $\theta \in [-\pi/2, \pi/2]$ $[\because x \in [-1, 1]]$

$\Rightarrow$ $-\pi/2 \le \theta \le \pi/2$

$\Rightarrow$ $-\pi/2 \le -\theta \le \pi/2$

$\Rightarrow$ $0 \le \frac{\pi}{2} - \theta \le \pi$

$\Rightarrow$ $\frac{\pi}{2} - \theta \in [0, \pi]$

Now, $\sin^{-1} x = \theta$

$\Rightarrow$ $x = \sin\theta$

$\Rightarrow$ $x = \cos\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow$ $\cos^{-1} x = \frac{\pi}{2} - \theta$

$\{\because x \in [-1, 1]$ and $(\pi/2 - \theta) \in [0, \pi])$

$\Rightarrow$ $\theta + \cos^{-1} x = \pi/2$ ... (ii)

from (i) and (ii), we get

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(ii) $\tan^{-1}x + \cot^{-1}x = \pi/2$, for all $x \in R$

Sol. Let, $\tan^{-1}x = \theta$ ... (i)

then, $\theta \in (-\pi/2, \pi/2)$ $\{\because x \in R)$

$\Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} < -\theta < \frac{\pi}{2}$

$\Rightarrow 0 < \frac{\pi}{2} - \theta < \pi$

$\Rightarrow \left(\frac{\pi}{2} - \theta\right) \in (0, \pi)$

Now, $\tan^{-1}x = \theta$

$\Rightarrow x = \tan\theta$

$\Rightarrow x = \cot(\pi/2 - \theta)$

$\Rightarrow \cot^{-1}x = \frac{\pi}{2} - \theta$ $\{\because \pi/2 - \theta \in (0, \pi)\}$

$\Rightarrow \theta + \cot^{-1}x = \frac{\pi}{2}$ ... (ii)

from (i) and (ii), we get

$\tan^{-1}x + \cot^{-1}x = \pi/2$

(iii) $\sec^{-1} + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$, for all $x \in (-\infty, -1] \cup [1, \infty)$

Sol. Let, $\sec^{-1}x = \theta$ ... (i)

then, $\theta \in [0, \pi] - \{\pi/2\}$ $\{\because x \in (-\infty, -1] \cup [1, \infty)\}$

$\Rightarrow 0 \le \theta \le \pi, \theta \ne \pi/2$

$\Rightarrow -\pi \le -\theta \le 0, \theta \ne \pi/2$

$\Rightarrow -\frac{\pi}{2} \le \frac{\pi}{2} - \theta \le \frac{\pi}{2}, \frac{\pi}{2} - \theta \ne 0$

$\Rightarrow \left(\frac{\pi}{2} - \theta\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \frac{\pi}{2} - \theta \ne 0$

Now, $\sec^{-1}x = \theta$

$\Rightarrow x = \sec\theta$

$\Rightarrow x = \operatorname{cosec}(\pi/2 - \theta)$

$\Rightarrow \operatorname{cosec}^{-1}x = \pi/2 - \theta$

$\left\{\because \left(\frac{\pi}{2} - \theta\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \frac{\pi}{2} - \theta \ne 0\right\}$

$\Rightarrow \theta + \operatorname{cosec}^{-1}x = \pi/2$ ... (ii)

from (i) and (ii); we get

$\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$

PROPERTY – II

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

Sol. Let, $\operatorname{cosec}^{-1}x = \theta$ ... (i)

then, $x = \operatorname{cosec}\theta$

$\Rightarrow \frac{1}{x} = \sin\theta$

$\{\because x \in (-\infty, -1] \cup [1, \infty) \Rightarrow \frac{1}{x} \in [-1, 1] \{0\}$

$\operatorname{cosec}^{-1}x = \theta \Rightarrow \theta \in [-\pi/2, \pi/2] - \{0\}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{x}\right)$ ... (ii)

from (i) and (ii); we get

$\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

Sol. Let, $\sec^{-1}x = \theta$ ... (i)

then, $x \in (-\infty, 1] \cup [1, \infty)$ and $\theta \in [0, \pi] - \{\pi/2\}$

Now, $\sec^{-1}x = \theta$

$\Rightarrow x = \sec\theta$

$\Rightarrow \frac{1}{x} = \cos\theta$

$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{x}\right)$ ... (ii)

$\left\{\begin{array}{l} \because x = (-\infty, -1] \cup [1, \infty) \\ \Rightarrow \frac{1}{x} \in [-1, 1] - \{0\} \text{ and } \theta \in [0, \pi] \end{array}\right.$

from (i) & (ii), we get

$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(x)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x \quad , & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$

Sol. Let $\cot^{-1} x = \theta$. Then $x \in R$, $x \neq 0$ and $\theta \in [0, \pi]$ ... (i)

Now two cases arises :

Case I : When $x > 0$

In this case, $\theta \in (0, \pi/2)$

$\therefore$ $\cot^{-1} x = \theta$

$\Rightarrow$ $x = \cot\theta$

$\Rightarrow$ $\frac{1}{x} = \tan\theta$

$\theta = \tan^{-1}\left(\frac{1}{x}\right)$ ... (ii)

from (i) and (ii), we get $\{\because \theta \in (0, \pi/2)\}$

$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$, for all $x > 0$.

Case II : When $x < 0$

In this case $\theta \in (\pi/2, \pi)$ $\{\because x = \cot\theta < 0\}$

Now, $\frac{\pi}{2} < \theta < \pi$

$\Rightarrow$ $-\frac{\pi}{2} < \theta - \pi < 0$

$\Rightarrow$ $\theta - \pi \in (-\pi/2, 0)$

$\therefore$ $\cot^{-1} x = \theta$

$\Rightarrow$ $x = \cot\theta$

$\Rightarrow$ $\frac{1}{x} = \tan\theta$

$\Rightarrow$ $\frac{1}{x} = -\tan(\pi - \theta)$

$\Rightarrow$ $\frac{1}{x} = \tan(\theta - \pi)$ $\{\because \tan(\pi - \theta) = -\tan\theta\}$

$\Rightarrow$ $\theta - \pi = \tan^{-1}\left(\frac{1}{x}\right)$ $\{\because \theta - \pi \in (-\pi/2, 0)\}$

$\Rightarrow$ $\tan^{-1}\left(\frac{1}{x}\right) = -\pi + \theta$ ... (iii)

from (i) and (iii), we get

$\tan^{-1}\left(\frac{1}{x}\right) = -\pi + \cot^{-1} x$, if $x < 0$

Hence,

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$$

PROPERTY – III

(i) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, for all $x \in [-1, 1]$

(ii) $\sec^{-1}, (-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, for all $x \in R$

(iv) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$

(v) $\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in R$

(vi) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

Sol. (ii) Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

let $\cos^{-1}(-x) = \theta$ ... (i)

then, $-x = \cos\theta$

$\Rightarrow$ $x = -\cos\theta$

$\Rightarrow$ $x = \cos(\pi - \theta)$

$\{\because x \in [-1, 1]$ and $\pi - \theta \in [0, \pi]$ for all $\theta \in [0, \pi]$

$\cos^{-1} x = \pi - \theta$

$\Rightarrow$ $\theta = \pi - \cos^{-1} x$ ... (ii)

from (i) and (ii), we get

$\cos^{-1}(-x) = \pi - \cos^{-1} x$

Similarly, we can prove other results.

(i) Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

let $\sin^{-1}(-x) = \theta$

then, $-x = \sin\theta$ ... (i)

$\Rightarrow$ $x = -\sin\theta$

$\Rightarrow$ $x = \sin(-\theta)$

$\Rightarrow$ $-\theta = \sin^{-1} x$

$\{\because x \in [-1, 1]$ and $-\theta \in [-\pi/2, \pi/2]$ for all $\theta \in [-\pi/2, \pi/2]$

$\Rightarrow$ $\theta = -\sin^{-1} x$ ... (ii)

from (i) and (ii), we get

$\sin^{-1}(-x) = -\sin^{-1}(x)$

PROPERTY – IV

(i) $\sin(\sin^{-1}x) = x$, for all $x \in [-1, 1]$

(ii) $\cos(\cos^{-1}x) = x$, for all $x \in [-1, 1]$

(iii) $\tan(\tan^{-1}x) = x$, for all $x \in R$

(iv) $\text{cosec}(\text{cosec}^{-1}x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec(\sec^{-1}x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot(\cot^{-1}x) = x$, for all $x \in R$

Sol. We know that, if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ exists such that $fof^{-1}(y) = f(f^{-1}(y)) = y$ for all $y \in B$.

Clearly, all these results are direct consequences of this property.

Aliter : Let $\theta \in [-\pi/2, \pi/2]$ and $x \in [-1, 1]$ such that $\sin\theta = x$.

then, $\theta = \sin^{-1}x$

$\therefore$ $x = \sin\theta = \sin(\sin^{-1}x)$

Hence, $\sin(\sin^{-1}x) = x$ for all $x \in [-1, 1]$

Similarly, we can prove other results.

Remark : It should be noted that,

$\sin^{-1}(\sin\theta) \neq \theta$, if $\notin [-\pi/2, \pi/2]$. Infact, we have

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi-\theta, & \text{if } \theta \in [-3\pi/2, -\pi/2] \\ \theta, & \text{if } \theta \in [-\pi/2, \pi/2] \\ \pi-\theta, & \text{if } \theta \in [\pi/2, 3\pi/2] \\ -2\pi+\theta, & \text{if } \theta \in [3\pi/2, 5\pi/2] \end{cases} \text{ and so on.}$$

Similarly,

$$\cos^{-1}(\cos\theta) = \begin{cases} -\theta, & \text{if } \theta \in [-\pi, 0] \\ \theta, & \text{if } \theta \in [0, \pi] \\ 2\pi-\theta, & \text{if } \theta \in [\pi, 2\pi] \\ -2\pi+\theta, & \text{if } \theta \in [2\pi, 3\pi] \end{cases} \text{ and so on.}$$

$$\tan^{-1}(\tan\theta) = \begin{cases} -\pi-\theta, & \text{if } \theta \in [-3\pi/2, -\pi/2] \\ \theta, & \text{if } \theta \in [-\pi/2, \pi/2] \\ \theta-\pi, & \text{if } \theta \in [\pi/2, 3\pi/2] \\ \theta-2\pi, & \text{if } \theta \in [3\pi/2, 5\pi/2] \end{cases} \text{ and so on.}$$

PROPERTY – V

(i) Sketch the graph for $y = \sin^{-1}(\sin x)$

Sol. As, $y = \sin^{-1}(\sin x)$ is periodic with period 2π.

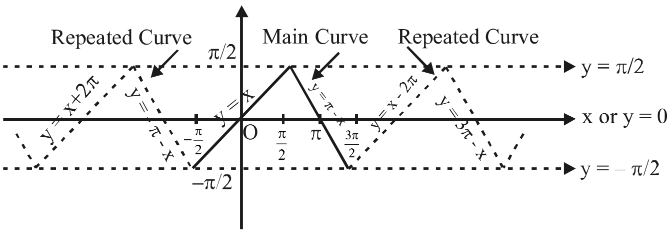
$\therefore$ to draw this graph we should draw the graph for one interval of length 2π and repeat for entire values of x.

As we know,

$$\sin^{-1}(\sin x) = \begin{cases} x; & -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\ (\pi - x); & -\dfrac{\pi}{2} \le \pi - x < \dfrac{\pi}{2}\left(\text{i.e., } \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}\right) \end{cases}$$

or $$\sin^{-1}(\sin x) = \begin{cases} x, & -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\ \pi - x, & \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}, \end{cases}$$

which is defined for the interval of length 2π, plotted as ;



Thus, the graph for $y = \sin^{-1}(\sin x)$, is a straight line up and a straight line down with slopes 1 and –1 respectively lying between $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

Note...

Students are adviced to learn the definition of $\sin^{-1}(\sin x)$ as,

$$y = \sin^{-1}(\sin x) = \begin{cases} x + 2\pi & ; -\dfrac{5\pi}{2} \le x \le -\dfrac{3\pi}{2} \\ -\pi - x & ; -\dfrac{3\pi}{2} \le x \le -\dfrac{\pi}{2} \\ x & ; -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\ \pi - x & ; \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2} \\ x - 2\pi & ; \dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2} \end{cases} \text{ ... and so on}$$

(ii) Sketch the graph for $y = \cos^{-1}(\cos x)$.

Sol. As, $y = \cos^{-1}(\cos x)$ is periodic with period 2π.

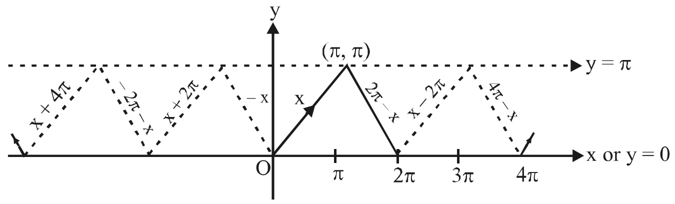
$\therefore$ to draw this graph we should draw the graph for one interval of length 2π and repear for entire values of x of length 2π.

As we know;

$$\cos^{-1}(\cos x) = \begin{cases} x; & 0 \le x \le \pi \\ 2\pi - x; & 0 \le 2\pi - x \le \pi, \end{cases}$$

or $$\cos^{-1}(\cos x) = \begin{cases} x; & 0 \le x \le \pi \\ 2\pi - x; & \pi \le x \le 2\pi, \end{cases}$$

Thus, it has been defined for $0 < x < 2\pi$ that has length 2π. So, its graph could be plotted as;



Thus, the curve $y = \cos^{-1}(\cos x)$.

(iii) Sketch the graph for $y = \tan^{-1}(\tan x)$.

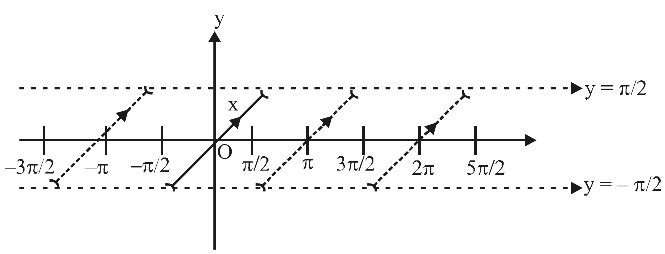
Sol. As $y = \tan^{-1}(\tan x)$ is periodic with period π.

$\frac{1}{2}$ $\therefore$ to draw this graph we should draw the graph for one interval of length π and repeat for entire values of x.

As we know; $\tan^{-1}(\tan x) = \left\{ x; -\frac{\pi}{2} < x < \frac{\pi}{2} \right\}$

Thus, it has been defined for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ that has length π.

So, its graph could be plotted as;



Thus, the curve for $y = \tan^{-1}(\tan x)$, where y is not defined for $x \in (2n+1)\frac{\pi}{2}$.

FORMULAS

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}, xy < 1$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}, xy > -1$

(iii) $2\tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}, |x| < 1$

(iv) $2\tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2}, |x| \le 1$

(v) $2\tan^{-1} x = \cos^{-1} \frac{1-x^2}{1+x^2}, x \ge 0$

(vi) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

(vii) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$

(viii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$

(ix) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2})$

(x) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if, $x > 0, y > 0, z > 0$ & $xy + yz + zx < 1$

Note :

(i) If $\tan^{-1} x + \tan^{-1}y + \tan^{-1}z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1} x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$ then $xy + yz + zx = 1$

REMEMBER THAT :

(i) $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2} \Rightarrow x = y = z = 1$

(ii) $\cos^{-1}x + \cos^{-1} y + \cos^{-1} z = 3\pi$ $x = y = z = -1$

(iii) $\tan^{-1} 1 + \tan^{-1} 2 + 2\tan^{-1} 3 =$

$\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{2}$