Welcome

To

MKS TUTORIALS

By

Manoj Sir

Topics covered in playlist of DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS: Rules for finding Complementary Functions, Rules for finding Particular Integrals, 5 most important problems on finding CF and PI, 4 most important problems on Method of Variations of Parameters, 4 most important problems on Cauchy's Homogeneous Linear Equations, 2 most important problems on Legendre's Linear Equations, 2 most important problems on Simultaneous Linear Equations.

Complete playlist of DIFFERENTIAL EQUATIONS of Higher Order with Constant Coefficient (in hindi): https://www.youtube.com/playlist?list=PLhSp9OSVmeyLTuT8bf-DZIffkRLmXmLzN

Complete playlist of DIFFERENTIAL EQUATIONS of Higher Order with Constant Coefficient (in english): https://www.youtube.com/playlist?list=PL0d0PH4hlFOfETGVOD9SrU9tiFg6m3UqT

Please Subscribe to our Hindi YouTube Channel–
MKS TUTORIALS: https://www.youtube.com/channel/UCbDs7CHAWVtyu81-6WIqZXg

Please Subscribe to our English YouTube Channel–
Manoj Sir TUTES: https://www.youtube.com/channel/UCj_4NZ5kRbWGbCn0VtSEmlQ

Please like my facebook page: https://www.facebook.com/MKS-Tutorials-2268483363380316/?modal=admin_todo_tour

LINEAR DIFFERENTIAL EQUATIONS

OR

DIFFERENTIAL EQUATION of HIGHER ORDER WITH CONSTANT CO-EFFICIENT

A] Rules for finding Complementary Functions
(4 different cases explained with examples)

B] Rules for finding Particular Integrals
(4 different cases + 1 specials case with examples)

C] Find CF and PI of:

Que (1). $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

Que (2) $\frac{d^2y}{dx^2} - 4y = x \sinh x$

Que (3) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^{x} \sin x$

Que (4) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

Que (5) $(D^4 + 2D^2 + 1)y = x^2 \cos x$.

D] Solve by Method of Variation of Parameters:

Que (1) $(D^2+4)y = \tan 2x$

Que (2) $y'' - 6y' + 9y = e^{3x}/x^2$

Que (3) $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

Que (4) $y'' - 2y' + y = e^{x} \log x$

E] Solve the CAUCHY'S HOMOGENEOUS LINEAR EQ.ⁿ

Que ① $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \log x$

Que ② $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = \log x \cdot \sin(\log x)$

Que ③ $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

Que ④ $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x)$

F] Solve the LEGENDRE'S LINEAR EQUATION

Que ① $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\log(1+x)$

Que ② $(2x-1)^2 \frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} - 2y = 8x^2 - 2x + 3$

G] Solve the SIMULTANEOUS LINEAR DIFFERENTIAL EQ.ⁿ

Que ① $\frac{dx}{dt} + y = \sin t \; ; \; \frac{dy}{dt} + x = \cos t$

Que ② $\frac{dx}{dt} + 2x - 3y = 5t$;

$\frac{dy}{dt} - 3x + 2y = 2e^{2t}$.

RULES FOR FINDING COMPLEMENTARY FUNCTIONS

To solve the eq.^{n} $\frac{d^n y}{dx^n} + \alpha_1 \frac{d^{n-1}y}{dx^{n-1}} + \alpha_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + \alpha_n y = 0$

where α's are constants.

The above eq.^{n} in symbolic form is

$$\left(D^n + \alpha_1 D^{n-1} + \alpha_2 D^{n-2} + \cdots + \alpha_n\right)y = 0$$

Its symbolic co-efficient is equated to zero.

i.e., $D^n + \alpha_1 D^{n-1} + \alpha_2 D^{n-2} + \cdots + \alpha_n = 0$

is called its Auxillary equation (AE).

$\left\{ \frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2, \quad \frac{d^3}{dx^3} = D^3 \right.$ and so on.

Let $m_1, m_2, m_3, \cdots m_n$ be its roots.

Case I: When the roots are real and distinct. (different)

i.e., $m_1, m_2, m_3, \cdots$

$$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots$$

Que. Solve: $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

Sol.^{n} The above eq.^{n} in symbolic form

$$\left(D^2 + 5D + 6\right)y = 0$$

Its AE is $m^2 + 5m + 6 = 0$

$\Rightarrow m^2 + 3m + 2m + 6 = 0 \Rightarrow m(m+3) + 2(m+3) = 0$

$\Rightarrow (m+3)(m+2) = 0 \Rightarrow m = -3, -2$

We get real and distinct roots.

$$\therefore CF = c_1 e^{-3x} + c_2 e^{-2x}$$

$\therefore$ The complete solution is

$$y = CF = c_1 e^{-3x} + c_2 e^{-2x}. \text{ Ans.}$$

Case II : When the roots are repeated.

i.e, $m_1, m_1, m_3, m_4, \cdots$

$$CF = (C_1 + xC_2)e^{m_1 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x} + \cdots$$

Que. Solve : $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Sol^n The above eq.^{n} in symbolic form

$$(D^2 + 6D + 9)y = 0$$

Its AE is $m^2 + 6m + 9 = 0 \Rightarrow (m+3)^2 = 0$

$\Rightarrow m = -3, -3.$

we get repeated roots.

$$\therefore CF = (C_1 + xC_2)e^{-3x}$$

$\therefore$ The complete solution is $y = CF = (C_1 + xC_2)e^{-3x}$.

Case III : When the roots are imaginary

$$m = \alpha \pm i\beta$$

$$CF = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$$

Que. Solve : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0$

Sol^n Symbolic form : $(D^2 - 2D + 4)y = 0$

Its AE is $m^2 - 2m + 4 = 0$

$$m = \frac{+2 \pm \sqrt{4 - 16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm i2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

we get imaginary roots.

$$CF = e^{x}(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x)$$

$\therefore$ The complete solution is

$$y = CF = e^{x}(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x)$$ Ans

Case IV: When two pairs of imaginary roots be equal.

i.e., $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha - i\beta$.

$$CF = e^{\alpha x}\left[(c_1 + xc_2)\cos\beta x + (c_3 + xc_4)\sin\beta x\right]$$

Que. Solve: $(D^4 + 2D^2 + 1)y = 0$

Sol.^{n} Its AE is

$$m^4 + 2m^2 + 1 = 0$$

$$\Rightarrow m^4 + m^2 + m^2 + 1 = 0$$

$$\Rightarrow m^2(m^2+1) + 1(m^2+1) = 0$$

$$\Rightarrow (m^2+1)(m^2+1) = 0$$

$$\Rightarrow m = \pm i, \pm i. \quad \alpha = 0, \beta = 1$$

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1}$

$= \pm i$

$$CF = e^{0.x}\left[(c_1 + xc_2)\cos x + (c_3 + xc_4)\sin x\right]$$

$$= (c_1 + xc_2)\cos x + (c_3 + xc_4)\sin x$$

$\therefore$ The complete solution is

$$y = CF = (c_1 + xc_2)\cos x + (c_3 + xc_4)\sin x.$$

Ans

RULES FOR FINDING PARTICULAR INTEGRAL

Consider the following eq.^{n}

$$\frac{d^n y}{dx^n} + \alpha_1 \frac{d^{n-1} y}{dx^{n-1}} + \alpha_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + \alpha_n y = X$$

$X \downarrow$ function of x.

which can be written in symbolic form:

$$\left(D^n + \alpha_1 D^{n-1} + \alpha_2 D^{n-2} + \cdots + \alpha_n\right) y = X$$

$$\therefore \quad PI = \frac{1}{D^n + \alpha_1 D^{n-1} + \alpha_2 D^{n-2} + \cdots + \alpha_n} \cdot X$$

$\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$ and so on.

Case I: When $X = e^{ax}$

$$\frac{1}{f(D)} \cdot e^{ax} = \frac{1}{f(a)} \cdot e^{ax} \quad \text{provided } f(a) \neq 0.$$

Que (1) Find PI of $(D^2 - 5D + 6) y = e^{4x}$.

Sol.^{n} $PI = \frac{1}{D^2 - 5D + 6} \cdot e^{4x} = \frac{1}{4^2 - (5 \times 4) + 6} \cdot e^{4x} = \frac{1}{2} \cdot e^{4x}$.

Case II: when $X = \sin(ax+b)$ or $\cos(ax+b)$

$$\frac{1}{f(D^2)} \cdot \sin(ax+b) = \frac{1}{f(-a^2)} \cdot \sin(ax+b) \; ; \text{ provided } f(-a^2) \neq 0.$$

Que (2). Find PI of $(D^2 - 5D + 6) y = \sin 3x$.

Sol.^{n} $PI = \frac{1}{D^2 - 5D + 6} \cdot \sin 3x = \frac{1}{-9 - 5D + 6} \cdot \sin 3x = \frac{-1}{5D+3} \cdot \sin 3x$

$$= -\frac{1}{(5D+3)} \times \left(\frac{5D-3}{5D-3}\right) \sin 3x = \frac{-(5D-3)}{25D^2 - 9} \sin 3x = \frac{-(5D-3)\sin 3x}{25(-9) - 9}$$

$$= \frac{\cancel{-}(5D-3)}{\cancel{-}234} \sin 3x = \frac{1}{234}\left[5 \cdot 3\cos 3x - 3\sin 3x\right]$$

$$= \frac{1}{234}\left(15 \cos 3x - 3 \sin 3x\right) \quad \text{Ans}$$

(2)

Que(3) Find PI of $(D^3+1)y = \cos(2x-1)$

Sol^n $PI = \dfrac{1}{D^2 \cdot D+1} \cdot \cos(2x-1) = \dfrac{1}{(-4)D+1} \cdot \cos(2x-1)$

$= \dfrac{1}{-4D+1} \cdot \cos(2x-1) = \dfrac{-1}{4D-1} \cos(2x-1)$

$= \dfrac{-1}{(4D-1)} \times \left(\dfrac{4D+1}{4D+1}\right) \cdot \cos(2x-1) = \dfrac{-(4D+1)}{16D^2-1} \cdot \cos(2x-1)$

$= \dfrac{-(4D+1)}{16(-4)-1} \cdot \cos(2x-1) = \dfrac{\cancel{-}(4D+1)}{\cancel{-}65} \cdot \cos(2x-1)$

$= \dfrac{1}{65}\left[4\{-\sin(2x-1)\} \cdot 2 + \cos(2x-1)\right]$

$= \dfrac{1}{65}\left[\cos(2x-1) - 8\sin(2x-1)\right]$ Ans

Case III: when $X = x^m$.

$PI = \dfrac{1}{f(D)} \cdot x^m = [f(D)]^{-1} \cdot x^m$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term D^m and operate on x^m term by term.

Que(4) Find PI of $(D^2+D)y = x^2+2x+4$.

Sol^n $PI = \dfrac{1}{D^2+D} \cdot (x^2+2x+4) = \dfrac{1}{D(D+1)}(x^2+2x+4)$

$$\boxed{\begin{array}{ll} \text{Use formulae:} & (1+D)^{-1} = 1-D+D^2-D^3+D^4-D^5+\cdots \\ & (1-D)^{-1} = 1+D+D^2+D^3+D^4+D^5+\cdots \end{array}}$$

$PI = \dfrac{1}{D}(1+D)^{-1}(x^2+2x+4) = \dfrac{1}{D}(1-D+D^2-\cdots)(x^2+2x+4)$

$= \dfrac{1}{D}(x^2+2\cancel{x}+4-\cancel{2x}-\cancel{2}+\cancel{2}) = \dfrac{1}{D}(x^2+4) = \int(x^2+4)\,dx$

$= \dfrac{x^3}{3}+4x$ Ans

Case IV : when $X = e^{ax}.V$, $V \rightarrow$ a function of x.

$$\frac{1}{f(D)} e^{ax}.V = e^{ax} \frac{1}{f(D+a)} . V$$

Que (5) Find PI of $(D^2 - 2D + 4)y = e^x \cos x$.

Sol^n $PI = \frac{1}{D^2 - 2D + 4} e^x . \cos x = e^x . \frac{1}{(D+1)^2 - 2(D+1) + 4} . \cos x$

$= e^x . \frac{1}{D^2 + \cancel{2D} + 1 - \cancel{2D} - 2 + 4} . \cos x = e^x . \frac{1}{D^2 + 3} . \cos x$

$= e^x . \frac{1}{-1+3} . \cos x = e^x . \frac{1}{2} . \cos x = \frac{e^x . \cos x}{2}$ Ans

SPECIAL CASE : when $f(a) = 0$

Que (6) Find PI of $(D^3 + 4D)y = \sin 2x$.

Sol^n $PI = \frac{1}{D^3 + 4D} . \sin 2x = \frac{1}{D(D^2 + 4)} . \sin 2x$

$= x . \frac{1}{3D^2 + 4} \sin 2x = x \frac{1}{3(-4) + 4} . \sin 2x$

$= -\frac{x}{8} \sin 2x$ Ans

Que (1) Solve: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

Sol.^{n} Symbolic form of above eq.^{n} is

$(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$

$\left\{ \frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2 \right.$

Its AE is $m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2.$

We get real and distinct roots.

$\therefore CF = c_1 e^{x} + c_2 e^{2x}$

Now, $PI = \frac{1}{D^2 - 3D + 2} \cdot (xe^{3x} + \sin 2x)$

$= \frac{1}{D^2 - 3D + 2} e^{3x} \cdot x + \frac{1}{D^2 - 3D + 2} \cdot \sin 2x$

$= e^{3x} \frac{1}{(D+3)^2 - 3(D+3) + 2} \cdot x + \frac{1}{-4 - 3D + 2} \cdot \sin 2x$

$= e^{3x} \cdot \frac{1}{D^2 + 6D + \cancel{9} - 3D - \cancel{9} + 2} \cdot x + \frac{1}{-2 - 3D} \sin 2x$

$= e^{3x} \cdot \frac{1}{D^2 + 3D + 2} \cdot x - \frac{1}{3D + 2} \sin 2x$

$= e^{3x} \frac{1}{2 + 3D + D^2} \cdot x - \frac{1}{(3D+2)} \times \frac{(3D-2)}{(3D-2)} \sin 2x$

$= e^{3x} \cdot \frac{1}{2\left(1 + \frac{3D}{2} + \frac{D^2}{2}\right)} \cdot x - \frac{3D - 2}{9D^2 - 4} \cdot \sin 2x$

$= \frac{e^{3x}}{2} \left[1 + \left(\frac{3D}{2} + \frac{D^2}{2}\right)\right]^{-1} \cdot x - \frac{(3D - 2)}{9(-4) - 4} \sin 2x$

$$= \frac{e^{3x}}{2}\left[1+\left(\frac{3D}{2}+\frac{D^2}{2}\right)\right]^{-1} \cdot x^1 - \frac{(3D-2)}{9(-4)-4}\sin 2x$$

$$\boxed{(1+D)^{-1} = 1 - D^1 + D^2 - D^3 + \cdots}$$

$$= \frac{e^{3x}}{2}\left[1-\left(\frac{3D}{2}+\frac{D^2}{2}\right)+\cdots\right]x + \frac{(3D-2)}{40}\sin 2x$$

$$= \frac{e^{3x}}{2}\left[x - \frac{3}{2}(1)\right] + \frac{1}{40}\left[3.2.\cos 2x - 2\sin 2x\right]$$

$$= \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right) + \frac{1}{40} \times 2\left(3\cos 2x - \sin 2x\right)$$

$$PI = \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right) + \frac{1}{20}\left(3\cos 2x - \sin 2x\right)$$

The complete solution is

$$y = CF + PI$$

$$= c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right) + \frac{1}{20}\left(3\cos 2x - \sin 2x\right)$$

Ans

Que 6) Solve: $\frac{d^2y}{dx^2} - 4y = x \sinh x$

Sol^n Symbolic form of given eq^n is

$(D^2-4)y = x \sinh x$

Its AE is $m^2-4=0 \Rightarrow m^2=4$.

$\Rightarrow m = \pm\sqrt{4} = \pm 2$.

$\frac{d}{dx} = D$

$\frac{d^2}{dx^2} = D^2$

we get real and distinct roots.

$\therefore CF = c_1 e^{2x} + c_2 e^{-2x}$.

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

Now, $PI = \frac{1}{D^2-4} \cdot x \sinh x$

$$= \frac{1}{D^2-4} x \left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left[\frac{1}{D^2-4} e^x \cdot x - \frac{1}{D^2-4} e^{-x} \cdot x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2-4} \cdot x - e^{-x} \frac{1}{(D-1)^2-4} \cdot x\right]$$

$$= \frac{1}{2}\left[e^x \cdot \frac{1}{D^2+2D-3} \cdot x - e^{-x} \frac{1}{D^2-2D-3} \cdot x\right]$$

$$= \frac{1}{2}\left[e^x \cdot \frac{1}{-3+2D+D^2} \cdot x - e^{-x} \frac{1}{-3-2D+D^2} \cdot x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3} \cdot \frac{1}{\left(1+\frac{2D}{-3}+\frac{D^2}{-3}\right)} \cdot x - \frac{e^{-x}}{-3} \cdot \frac{1}{1-\frac{2D}{-3}+\frac{D^2}{-3}} \cdot x\right]$$

$$= \frac{1}{2}\left[-\frac{e^x}{3} \cdot \frac{1}{\left(1-\frac{2D}{3}-\frac{D^2}{3}\right)} \cdot x + \frac{e^{-x}}{3} \cdot \frac{1}{1+\frac{2D}{3}-\frac{D^2}{3}} \cdot x\right]$$

$$= \frac{1}{2}\left[-\frac{e^x}{3}\underbrace{\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1}}_{(1-D)^{-1}} \cdot x + \frac{e^{-x}}{3}\underbrace{\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1}}_{(1+D)^{-1}} \cdot x\right]$$

$$= \frac{1}{2}\left[-\frac{e^{x}}{3}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1} \cdot x^1 + \frac{e^{-x}}{3}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1} \cdot x^1\right.$$

$$\boxed{\begin{aligned}(1-D)^{-1} &= 1+D+D^2+D^3+\cdots\\ (1+D)^{-1} &= 1-D+D^2-D^3+\cdots\end{aligned}}$$

$$= \frac{1}{2}\left[-\frac{e^{x}}{3}\left\{1+\frac{2D}{3}+\frac{D^2}{3}+\cdots\right\}x + \frac{e^{-x}}{3}\left\{1-\frac{2D}{3}+\frac{D^2}{3}+\cdots\right\}x^1\right]$$

$$= \frac{1}{2}\left[-\frac{e^{x}}{3}\left(x+\frac{2}{3}\right)+\frac{e^{-x}}{3}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{e^{x}\cdot x}{6} - \frac{e^{x}}{9} + \frac{e^{-x}}{6}x - \frac{e^{-x}}{9}$$

$$= \frac{x}{6}\left(-e^{x}+e^{-x}\right) - \frac{1}{9}\left(e^{x}+e^{-x}\right)$$

$$= -\frac{x}{3}\left(\frac{e^{x}-e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$\text{I.} = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

The complete solution is

$$\begin{aligned} y &= CF + PI\\ &= c_1 e^{2x} + c_2 e^{-2x} + \left(-\frac{x}{3}\sinh x - \frac{2}{9}\cosh x\right)\\ &= c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x \end{aligned}$$

Ans.

Que③ Solve: $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin x.$

Sol.ⁿ Symbolic form of given eq.ⁿ is

$$(D^2 - 2D + 1)\,y = xe^x \sin x.$$

$$\left\{ \frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2 \right.$$

Its AE is $m^2 - 2m + 1 = 0.$

$\Rightarrow (m-1)^2 = 0. \Rightarrow m = 1, 1.$

we get repeated roots.

$\therefore \quad CF = (C_1 + xC_2)e^x$

$$PI = \frac{1}{D^2 - 2D + 1}.(x e^x \sin x) = e^x \frac{1}{(D+1)^2 - 2(D+1) + 1}.x \sin x$$

$$= e^x \frac{1}{D^2 + \cancel{2D} + \cancel{1} - \cancel{2D} - 2 + 1}.x \sin x = e^x . \frac{1}{D^2}(x \sin x)$$

$$= e^x . \frac{1}{D} . \int \underset{I}{x} \underset{II}{\sin x}\, dx \qquad \text{ILATE}$$

$$= e^x . \frac{1}{D} . \left[x(-\cos x) - \int 1.(-\cos x)\, dx \right]$$

$$= e^x . \frac{1}{D} \left[-x\cos x + \int \cos x\, dx \right]$$

$$= e^x . \frac{1}{D} (-x\cos x + \sin x)$$

$$= e^x \left[\int \sin x\, dx - \int \underset{I}{x} \underset{II}{\cos x}\, dx \right]$$

$$= e^x \left[-\cos x - \left\{ x(\sin x) - \int 1.(\sin x)\, dx \right\} \right]$$

$$= e^x \left[-\cos x - x\sin x + (-\cos x) \right]$$

$$= -e^x (2\cos x + x\sin x) = PI.$$

The complete solution is

$$y = CF + PI$$
$$= (C_1 + x C_2) e^x - e^x (2\cos x + x \sin x)$$

Ans

Que (4) Solve: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

Sol.^{n} Symbolic form of above eq.^{n} is

$$(D^2 + 3D + 2)y = e^{e^x}$$

$\left\{ \frac{d}{dx} = D,\ \frac{d^2}{dx^2} = D^2 \right.$

Its AE is $m^2 + 3m + 2 = 0$

$\Rightarrow m^2 + 2m + m + 2 = 0$

$\Rightarrow m(m+2) + 1(m+2) = 0$

$\Rightarrow (m+2)(m+1) = 0 \quad \Rightarrow m = -2, -1$

$\therefore CF = C_1 e^{-2x} + C_2 e^{-x}$ Roots are real and distinct.

Now, $PI = \frac{1}{D^2 + 3D + 2} \cdot e^{e^x} = \frac{1}{(D+2)(D+1)} e^{e^x}$.

Use formula: $\boxed{\frac{1}{D-a} \cdot Q = e^{ax} \int Q \cdot e^{-ax}\, dx}$

$$PI = \frac{1}{D+2}\left[\frac{1}{D+1} \cdot e^{e^x}\right] = \frac{1}{D+2}\left[\frac{1}{D-(-1)} e^{e^x}\right]$$

$$= \left(\frac{1}{D+2}\right)\left[e^{-x} \int e^{e^x} \cdot e^x\, dx\right]$$

Put $e^x = t \Rightarrow e^x dx = dt$

$$PI = \left(\frac{1}{D+2}\right)\left[e^{-x} \int e^t\, dt\right] = \left(\frac{1}{D+2}\right)\left(e^{-x} e^t\right)$$

$$= \left(\frac{1}{D+2}\right) e^{-x} \cdot e^{e^x} = \frac{1}{D-(-2)} e^{-x} \cdot e^{e^x}$$

$$= e^{-2x} \int e^{-x} \cdot e^{e^x} \cdot e^{2x}\, dx = e^{-2x} \int e^{e^x} e^x\, dx$$

Put $e^x = t \Rightarrow e^x dx = dt$

$$PI = e^{-2x} \int e^t\, dt = e^{-2x} \cdot e^t = \underline{e^{-2x} \cdot e^{e^x}}$$

The complete solution is

$$y = CF + PI$$
$$= C_1 e^{-2x} + C_2 e^{-x} + e^{-2x} \cdot e^{e^x}$$

Ans.

Que 5 Solve: $(D^4+2D^2+1)y = x^2\cos x$

Sol^n Its AE is $m^4+2m^2+1=0$ $\qquad m^2=-1$

$(m^2+1)^2=0 \qquad m=\pm\sqrt{-1}=\pm i$

$m=\pm i, \pm i$

$CF = (C_1 + xC_2)\cos x + (C_3 + xC_4)\sin x. \qquad \boxed{e^{ix} = \cos x + i\sin x}$

$$PI = \frac{1}{D^4+2D^2+1}\cdot x^2\cos x = \frac{1}{(D^2+1)^2}\,(\text{Re.P. of } e^{ix})\cdot x^2$$

$$= (\text{Re.P. of } e^{ix})\frac{1}{[(D+i)^2+1]^2}\cdot x^2 = (\text{Re.P. of } e^{ix})\frac{1}{(D^2+2iD+\cancel{i^2}+\cancel{1})^2}\cdot x^2$$

$$= (\text{Re.P. of } e^{ix})\frac{1}{(D^2+2iD)^2}\cdot x^2 = (\text{Re.P. of } e^{ix})\frac{1}{4i^2D^2}\,\frac{1}{\left(\frac{D^{\cancel{2}}}{2i\cancel{D}}+1\right)^2}\cdot x^2$$

$$= (\text{Re.P. of } e^{ix})\left(\frac{-1}{4D^2}\right)\frac{1}{\left(1+\frac{D}{2i}\right)^2}\cdot x^2$$

$$= \text{Re.P. of } e^{ix}\left(\frac{-1}{4D^2}\right)\frac{1}{\left(1-\frac{i^{\cancel{2}}D}{2\cancel{i}}\right)^2}\cdot x^2$$

$$= \text{Re.P. of } e^{ix}\left(-\frac{1}{4D^2}\right)\left(1-\frac{iD}{2}\right)^{-2}\cdot x^2$$

Use formula:

$\Rightarrow \quad (1+D)^{-2} = 1-2D+3D^2-4D^3+5D^4-6D^5+\cdots$

$(1-D)^{-2} = 1+2D+3D^2+4D^3+5D^4+6D^5+\cdots$

$$= \left(-\frac{1}{4}\right)(\text{Re.P. of } e^{ix})\cdot\frac{1}{D^2}\left[1+\cancel{2}\cdot i\frac{D}{\cancel{2}}+3\left(\frac{iD}{2}\right)^2+\cdots\right]x^2$$

$$= \left(-\frac{1}{4}\right)(\text{Re.P. of } e^{ix})\,\frac{1}{D^2}\left[x^2+i2x+\frac{3}{4}(-1)(2)\right]$$

$$= \left(-\frac{1}{4}\right)\left(\text{Re. P. of } e^{ix}\right)\frac{1}{D^2}\left[x^2 + i2x + \frac{3}{4}(-1)(2)\right]$$

$$= \left(-\frac{1}{4}\right)\left(\text{Re. P. of } e^{ix}\right)\frac{1}{D^2}\left[x^2 + i2x - \frac{3}{2}\right]$$

$$= \left(-\frac{1}{4}\right)\text{Re. P. of } e^{ix}\cdot\frac{1}{D}\cdot\left(\frac{x^3}{3} + i\cancel{2}\frac{x^2}{\cancel{2}} - \frac{3}{2}x\right)$$

$$= \left(-\frac{1}{4}\right)\text{Re. P. of } e^{ix}\cdot\left(\frac{x^4}{12} + i\frac{x^3}{3} - \frac{3}{4}x^2\right)$$

$$= \left(-\frac{1}{4}\right)\text{Re. P. of } (\cos x + i\sin x)\left(\frac{x^4}{12} - \frac{3}{4}x^2 + i\frac{x^3}{3}\right)$$

$$= \left(-\frac{1}{4}\right)\text{Re. P. of }\left(\cos x\cdot\frac{x^4}{12} - \cos x\cdot\frac{3}{4}x^2 + i\cos x\cdot\frac{x^3}{3} + i\sin x\cdot\frac{x^4}{12} - i\sin x\cdot\frac{3}{4}x^2 + i^2\sin x\cdot\frac{x^3}{3}\right)$$

$$= \left(-\frac{1}{4}\right)\text{Re. P. of}\left(\frac{x^4}{12}\cos x - \frac{3}{4}x^2\cos x + i\frac{x^3}{3}\cos x + i\frac{x^4}{12}\sin x - i\cdot\frac{3}{4}x^2\cdot\sin x - \frac{x^3}{3}\sin x\right)$$

$$PI = \left(-\frac{1}{4}\right)\left(\frac{x^4}{12}\cos x - \frac{3}{4}x^2\cos x - \frac{x^3}{3}\sin x\right)$$

$\therefore$ The complete solution is

$$y = CF + PI$$

$$= (C_1 + xC_2)\cos x + (C_3 + xC_4)\sin x - \frac{1}{4}\left(\frac{x^4}{12}\cos x - \frac{3}{4}x^2\cos x - \frac{x^3}{3}\sin x\right)$$

Ans

Que (1) Solve by Method of Variation of parameters:

$$(D^2+4)y = \tan 2x$$

Sol.ⁿ Its AE is $m^2+4=0$

$$\Rightarrow m^2=-4 \Rightarrow m = \pm\sqrt{-4} = \pm\sqrt{-1}\sqrt{4}$$

$$m = \pm i2 = 0 \pm i2$$

$$CF = e^{0.x}(C_1\cos 2x + C_2 \sin 2x) = C_1 \cos 2x + C_2 \sin 2x$$

$$PI = -y_1 \int \frac{y_2 \cdot X}{W} dx + y_2 \int \frac{y_1 \cdot X}{W} dx$$

$y_1 = \cos 2x$; $y_2 = \sin 2x$; $X = \tan 2x$; $\cancel{W}$

Wronskian determinant,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x = 2(\cos^2 2x + \sin^2 2x) = 2.$$

$$PI = -\cos 2x \int \frac{\sin 2x \cdot \tan 2x}{2} dx + \sin 2x \int \frac{\cos 2x \cdot \tan 2x}{2} dx$$

$$= -\cos 2x \int \frac{\sin^2 2x}{2\cos 2x} dx + \sin 2x \int \frac{\sin 2x}{2} dx$$

$$= -\cos 2x \int \left(\frac{1-\cos^2 2x}{2\cos 2x}\right) dx + \frac{\sin 2x}{2}\left(-\frac{\cos 2x}{2}\right)$$

$$= -\frac{\cos 2x}{2}\int (\sec 2x - \cos 2x)dx - \frac{\sin 2x \cdot \cos 2x}{4}$$

$$= -\frac{\cos 2x}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right] - 2\frac{\sin 2x \cos 2x}{8}$$

$$= -\frac{\cos 2x}{4}\left[\log(\sec 2x + \tan 2x)\right] + \frac{\cancel{\cos 2x \sin 2x}}{4}$$

$$\boxed{PI = -\frac{\cos 2x}{4}\log(\sec 2x + \tan 2x)} - \frac{\cancel{\sin 2x \cos 2x}}{4}$$

$\therefore$ The complete solution is

$$y = CF + PI$$

$$= C_1 \cos 2x + C_2 \sin 2x - \frac{\cos 2x}{4} \log(\sec 2x + \tan 2x)$$

Ans

Que ② Solve by Method of variation of parameters:

$$y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$$

Sol.ⁿ

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$$

Symbolic form : $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

Its AE is $m^2 - 6m + 9 = 0$

$$\Rightarrow (m-3)^2 = 0 \Rightarrow m = 3, 3$$

Roots are repeated. $\therefore CF = (C_1 + xC_2)e^{3x} = C_1 \cdot e^{3x} + C_2 x e^{3x}$

$y_1 = e^{3x}$, $y_2 = xe^{3x}$, $X = \frac{e^{3x}}{x^2}$

$y_1' = 3e^{3x}$; $y_2' = x\,3e^{3x} + e^{3x}$

$$= e^{3x}(3x+1)$$

Wronskian Determinant,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}(3x+1) \end{vmatrix}$$

$$= e^{6x}(3x+1) - 3x\,e^{6x}$$

$$= \cancel{e^{6x}\cdot 3x} + e^{6x} - \cancel{3xe^{6x}} = \boxed{e^{6x} = W}$$

$$PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 \cdot X}{W} dx$$

$$= -e^{3x} \int \frac{x\cancel{e^{3x}}}{\cancel{e^{6x}}} \cdot \frac{\cancel{e^{3x}}}{x^2} dx + xe^{3x} \int \frac{\cancel{e^{3x}}}{\cancel{e^{6x}}} \cdot \frac{\cancel{e^{3x}}}{x^2} dx$$

$$= -e^{3x} \int \frac{dx}{x} + xe^{3x} \int x^{-2} dx$$

$$= -e^{3x} \log x + xe^{3x}\left(\frac{x^{-1}}{-1}\right) = -e^{3x}\log x - \cancel{x}e^{3x} \cdot \frac{1}{\cancel{x}}$$

$$= -e^{3x}(\log x + 1)$$

The complete solution is

$$y = CF + PI$$
$$= (c_1 + x c_2) e^{3x} - e^{3x} (\log x + 1)$$

Ans.

Que(3) Solve by method of variation of parameters:

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

Sol^n Given eq.^{n} in symbolic form: $(D^2-1)y = \frac{2}{1+e^x}$

Its AE is $m^2-1=0 \Rightarrow m^2=1 \Rightarrow m=\pm 1$.

$\therefore CF = c_1e^x + c_2e^{-x}$. Roots are real and distinct

Now, $PI = -y_1\int \frac{y_2 X}{W}dx + y_2\int \frac{y_1 \cdot X}{W}dx$

Here, $y_1 = e^x$; $y_2 = e^{-x}$; $X = \frac{2}{1+e^x}$

$y_1' = e^x$; $y_2' = -e^{-x}$

Wronskian Determinant, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$

$W = -e^x \cdot e^{-x} - e^x \cdot e^{-x} = -1-1 = -2$.

$$\therefore PI = -e^x\int \frac{e^{-x}}{-2}\cdot\frac{2}{1+e^x}dx + e^{-x}\int\frac{e^x}{-2}\cdot\frac{2}{1+e^x}dx$$

$$= e^x\int\frac{dx}{e^x(1+e^x)} - e^{-x}\int\frac{e^x}{1+e^x}dx$$

$$= e^x\int\frac{dx}{e^x(1+e^x)} - e^{-x}\log(1+e^x) \quad \text{——(1)}$$

Solving, $\frac{1}{e^x(1+e^x)} = \frac{A}{e^x} + \frac{B}{1+e^x} = \frac{A(1+e^x)+Be^x}{e^x(1+e^x)}$

$\Rightarrow 1 = A(1+e^x) + Be^x$.

Put $x = -\infty \Rightarrow 1 = A(1+0) + B\cdot 0 \Rightarrow A = 1$

Put $x = 0 \Rightarrow 1 = A(2) + B(1) \Rightarrow 2A + B = 1$

$\Rightarrow B = -1$

$$\therefore \frac{1}{e^x(1+e^x)} = \frac{1}{e^x} - \frac{1}{1+e^x}$$

$\therefore$ from (1),

$$PI = e^x \int \left(\frac{1}{e^x} - \frac{1}{1+e^x} \right) dx - e^{-x} \log(1+e^x)$$

$$= e^x \left[\int e^{-x} dx - \int \frac{e^{-x}}{e^{-x}} \left(\frac{1}{1+e^x} \right) dx \right] - e^{-x} \log(1+e^x)$$

$$= e^x \left[\frac{e^{-x}}{-1} - \int \frac{e^{-x}}{e^{-x}+1} dx \right] - e^{-x} \log(1+e^x)$$

$$= e^x \left[-e^{-x} + \log(e^{-x}+1) \right] - e^{-x} \log(1+e^x)$$

$$PI = -1 + e^x \log(e^{-x}+1) - e^{-x} \log(1+e^x)$$

$\therefore$ The complete solution is

$$y = CF + PI$$

$$= c_1 e^x + c_2 e^{-x} - 1 + e^x \log(e^{-x}+1) - e^{-x} \log(e^x+1)$$

Ans.

Que (4) Solve by method of variation of parameters: $y'' - 2y' + y = e^x \log x$

Sol.^{n} Given: $y'' - 2y' + y = e^x \log x$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \log x$$

Symbolic form: $(D^2 - 2D + 1)y = e^x \log x$.

Its AE is $m^2 - 2m + 1 = 0$

$$\Rightarrow (m-1)^2 = 0 \quad \Rightarrow m = 1, 1$$

Roots are repeated.

Hence, $CF = (c_1 + x c_2) e^x = e^x c_1 + x e^x c_2$

Now, $PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

Here, $y_1 = e^x$; $y_2 = xe^x$; $X = e^x \log x$

$y_1' = e^x$; $y_2' = (x+1)e^x$

$$\therefore PI = -e^x \int \frac{xe^x \cdot e^x}{e^{2x}} \log x \, dx + xe^x \int \frac{e^x \cdot e^x \log x}{e^{2x}} dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (x+1)e^{2x} - xe^{2x}$$
$$= xe^{2x} + e^{2x} - xe^{2x}$$
$$= e^{2x}$$

$$PI = -e^x \int x \log x \, dx + xe^x \int \log x \, dx$$
$$= -e^x \left[\log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right] + xe^x \left[\log x \cdot x - \int \frac{1}{x} \cdot x \, dx \right]$$
$$= -e^x \left[\frac{x^2}{2} \log x - \frac{x^2}{4} \right] + xe^x \left[x \log x - x \right]$$
$$= -\frac{e^x \cdot x^2}{2} \log x + \frac{e^x \cdot x^2}{4} + x^2 e^x \log x - x^2 e^x.$$

$$PI = x^2 e^x \log x\left(1-\frac{1}{2}\right) + x^2 e^x\left(\frac{1}{4}-1\right)$$

$$= \frac{1}{2} x^2 e^x \log x - \frac{3}{4} x^2 e^x$$

$$PI = x^2 e^x\left(\frac{\log x}{2} - \frac{3}{4}\right)$$

The complete solution is

$$y = CF + PI$$

$$= (c_1 + x c_2)e^x + x^2 e^x\left(\frac{\log x}{2} - \frac{3}{4}\right)$$

Ans.

CAUCHY'S HOMOGENEOUS LINEAR EQUATION

To solve the eq.^{n}

$$x^n \frac{d^n y}{dx^n} + \alpha_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \alpha_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + \alpha_n y = X \quad \text{(i)}$$

is called Cauchy's homogeneous linear eq.^{n}
where, X is a function of x,
and, α's are constants.

To reduce such eq.^{ns} to linear differential eq.^{ns} with constant co-efficients,

Put $x = e^t \Rightarrow t = \log x$

$\log_e x = \log_e e^t = t \log_e e$
$\Rightarrow \log x = t$

$x \frac{dy}{dx} = Dy$, $x^2 \frac{d^2 y}{dx^2} = D(D-1)y$, and so on.

where, $D = \frac{d}{dt}$ $\qquad x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$

Substituting these values in eq.^{n} (i), then solve it as before, we did in linear differential eq.^{n} with constant co-efficients.

Que (1) Solve: $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Sol^n Put $x = e^t \Rightarrow t = \log x$

$\therefore x\frac{dy}{dx} = Dy$; $x^2\frac{d^2y}{dx^2} = D(D-1)y$ where $D = \frac{d}{dt}$

$\therefore$ The eq.^{n} becomes,

$[D(D-1) - D + 1]y = t$

Its AE is $m(m-1) - m + 1 = 0$

$\Rightarrow m^2 - m - m + 1 = 0 \Rightarrow (m-1)^2 = 0$

$\Rightarrow m = 1, 1$ (Repeated roots)

$\therefore CF = (C_1 + tC_2)e^t$

Now, $PI = \frac{1}{(D-1)^2}\cdot t = \frac{1}{(-1)^2(1-D)^2}\cdot t = (1-D)^{-2}\cdot t$

Use formula: $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + 5D^4 + \cdots$

$\therefore PI = (1 + 2D + \cdots)t = t + 2.$

$\therefore$ The complete solution is

$y = CF + PI$

$= (C_1 + tC_2)e^t + t + 2$

$= (C_1 + \log x \cdot C_2)x + \log x + 2$

Ans

Que (2) Solve: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \cdot \sin(\log x)$

Sol.^{n} Put $x = e^t \Rightarrow \log_e x = t$

$$x\frac{dy}{dx} = Dy \;;\; x^2\frac{d^2y}{dx^2} = D(D-1)y \quad \text{where } D = \frac{d}{dt}$$

$\therefore$ The eq.^{n} becomes;

$$[D(D-1) + D + 1]y = t.\sin t$$

$$\Rightarrow (D^2 - D + D + 1)y = t\sin t$$

$$\Rightarrow (D^2+1)y = t\sin t$$

Its AE is $m^2+1=0$

$$m^2 = -1 \Rightarrow m = \pm\sqrt{-1} = 0 \pm i1$$

$$\therefore CF = C_1\cos t + C_2\sin t \quad \text{(Imaginary roots)}$$

Now, $PI = \frac{1}{D^2+1}.t\sin t$ $\boxed{\because e^{it} = \cos t + i\sin t}$

$$= \frac{1}{D^2+1}(\text{I.P. of } e^{it}).t = \text{IP of } e^{it}.\frac{1}{(D+i)^2+1}.t$$

$$= \text{IP. of } e^{it}.\frac{1}{D^2+2iD+i^2+1}.t = \text{IP. of } e^{it}.\frac{1}{2iD+D^2}.t$$

$$= \text{IP of } e^{it}.\frac{1}{2iD}.\frac{1}{\left(1+\frac{D^2}{2iD}\right)}.t$$

$$= \text{I.P. of } e^{it}.\frac{1}{2iD}.\frac{1}{\left(1-\frac{i^2D^2}{2iD}\right)}.t$$

$$= \text{IP of } e^{it}\cdot\frac{1}{2iD}\cdot\left(1-\frac{iD}{2}\right)^{-1}\cdot t$$

$(1-D)^{-1} = 1+D+D^2+D^3$

$$= \text{IP of } e^{it}\cdot\frac{1}{2iD}\left(1+\frac{iD}{2}+\cdots\right)t$$

$$= \text{IP of } e^{it}\cdot\frac{1}{2iD}\left(t+\frac{i}{2}\right)$$

$$= \text{IP of } e^{it}\cdot\frac{1}{2i}\int\left(t+\frac{i}{2}\right)dt$$

$$= \text{IP of } e^{it}\cdot\frac{1}{2i}\left(\frac{t^2}{2}+\frac{it}{2}\right)$$

$$= \text{IP of } e^{it}\cdot\left(\frac{t^2}{4i}+\frac{it}{4i}\right)$$

$$= \text{IP of } e^{it}\left(-\frac{i^2t^2}{4i}+\frac{t}{4}\right) = \text{IP of } e^{it}\left(\frac{t}{4}-\frac{it^2}{4}\right)$$

$$= \text{IP of } (\cos t + i\sin t)\left(\frac{t}{4}-i\frac{t^2}{4}\right)$$

$$= \text{IP of } \left[\frac{t}{4}\cos t - i\cos t\cdot\frac{t^2}{4} + i\frac{t}{4}\sin t - i^2\frac{t^2}{4}\cdot\sin t\right]$$

$$= \text{IP of } \left(\frac{t}{4}\cos t - i\frac{t^2}{4}\cos t + i\frac{t}{4}\sin t + \frac{t^2}{4}\sin t\right)$$

$$PI = -\frac{t^2}{4}\cos t + \frac{t}{4}\sin t$$

$\therefore$ The complete solution is

$$y = CF + PI$$

$$= c_1\cos t + c_2\sin t - \frac{t^2}{4}\cos t + \frac{t}{4}\sin t$$

$$= c_1\cos\log x + c_2\sin\log x - \frac{1}{4}(\log x)^2\cos(\log x) + \frac{1}{4}\log x\,\sin(\log x)$$

Ans.

Que (3) Solve: $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x+\frac{1}{x}\right)$

Sol^n Put $x = e^t \Rightarrow t = \log_e x$

$x\frac{dy}{dx} = Dy$; $x^2\frac{d^2y}{dx^2} = D(D-1)y$; $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

The eq^n becomes, where, $D = \frac{d}{dt}$

$\left[D(D-1)(D-2) + 2D(D-1) + 2\right]y = 10\left(e^t + \frac{1}{e^t}\right)$

$\Rightarrow \left(D^3 - D^2 + 2\right)y = 10\left(e^t + e^{-t}\right)$

Its AE is $m^3 - m^2 + 2 = 0 \Rightarrow m = -1, 1\pm i$

$\therefore$ CF $= c_1e^{-t} + e^t(c_2\cos t + c_3\sin t)$

Now, PI $= \frac{1}{D^3-D^2+2}\cdot 10(e^t + e^{-t})$

$= \frac{10}{D^3-D^2+2}e^t + \frac{10}{D^3-D^2+2}e^{-t} = \frac{10}{1-1+2}e^t + \frac{10t}{3D^2-2D}\cdot e^{-t}$

$= 5e^t + 10t\frac{1}{3(-1)^2-2(-1)}\cdot e^{-t} = 5e^t + \frac{10t}{5}e^{-t}$

PI $= 5e^t + 2te^{-t}$

The complete solution is

$y = $ CF + PI

$= c_1e^{-t} + e^t(c_2\cos t + c_3\sin t) + 5e^t + 2te^{-t}$

$= c_1x^{-1} + x(c_2\cos\log x + c_3\sin\log x) + 5x + 2\log x\cdot x^{-1}$

Ans

Que (4) Solve: $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos\log x$

Sol.^{n} Put $x = e^t \Rightarrow t = \log_e x$

$x\frac{dy}{dx} = Dy$; $x^2\frac{d^2y}{dx^2} = D(D-1)y$; $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

The eq.^{n} becomes, where, $D = \frac{d}{dt}$

$$[D(D-1)(D-2) + 3D(D-1) + D + 8]y = 65\cos t$$

$$\Rightarrow \quad (D^3+8)y = 65\cos t$$

Its AE is $m^3 + 8 = 0 \Rightarrow m = -2,\ 1 \pm i\sqrt{3}$

$$CF = c_1e^{-2t} + e^t(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t)$$

Now, $PI = \frac{1}{D^3+8}\,65\cos t = 65\cdot\frac{1}{D\cdot D^2+8}\cdot\cos t$

$$= 65\,\frac{1}{D(-1^2)+8}\cos t = 65\,\frac{1}{-D+8}\cos t = -65\,\frac{1}{D-8}\cos t$$

$$= -65\cdot\frac{1}{D-8}\times\frac{D+8}{D+8}\cdot\cos t = -65\,\frac{D+8}{D^2-64}\cos t$$

$$= -65\cdot\frac{D+8}{-1^2-64}\cdot\cos t = -65\cdot\frac{D+8}{-65}\cos t$$

$$= -\sin t + 8\cos t = 8\cos t - \sin t = PI$$

The complete solution is

$$y = CF + PI$$

$$= c_1e^{-2t} + e^t(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t)$$

$$+ 8\cos t - \sin t$$

$$= c_1e^{-2\log x} + x(c_2\cos\sqrt{3}\log x + c_3\sin\sqrt{3}\log x)$$

$$+ 8\cos\log x - \sin\log x$$

Ans.

LEGENDRE'S LINEAR EQUATION

Equation of form

$$(ax+b)^n \frac{d^n y}{dx^n} + \alpha_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + \alpha_n y = X \qquad \text{—— (1)}$$

is called Legendre's linear equation.
where X is a function of x and α's are constants.

To reduce the above eq.^{n} in linear differential equation, take substitution:

$$ax+b = e^t \qquad \log_e(ax+b) = \log_e e^t = t \log_e e = t$$

$$\Rightarrow \log_e(ax+b) = t$$

$$(ax+b)\frac{dy}{dx} = a\,Dy \;;\; (ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y \;;$$

$$(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y \text{ and so on}$$

where, $D = \frac{d}{dt}$

Substituting these values in eq.^{n} (1), then solve it as before we performed in linear differential eq.^{n} with constant co-efficients.

Que ① Solve: $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\log(1+x)$

Sol.ⁿ The given eq.ⁿ is a legendre's linear equation.

Put $(1+x) = e^t \Rightarrow t = \log_e(1+x)$

$(1+x)\frac{dy}{dx} = Dy$; $(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$ where, $D = \frac{d}{dt}$

The eq.ⁿ becomes,

$$[D(D-1) + D + 1]y = 2\sin t$$

$$\Rightarrow (D^2 - D + D + 1)y = 2\sin t$$

$$\Rightarrow (D^2+1)y = 2\sin t$$

Its AE is $m^2+1=0 \Rightarrow m^2 = -1$

$$\Rightarrow m = \pm\sqrt{-1} = \pm i \text{ (Imaginary roots)}$$

$$\therefore CF = c_1\cos t + c_2\sin t$$

Now, $PI = \frac{1}{D^2+1} \cdot 2\sin t = 2t\frac{1}{2D}\cdot\sin t$

$$= t\int \sin t\,dt = t(-\cos t) = -t\cos t$$

The complete solution is

$$y = CF + PI$$

$$= c_1\cos t + c_2\sin t - t\cos t$$

$$= c_1\cos\log(1+x) + c_2\sin\log(1+x)$$

$$- \log(1+x)\cdot\cos\log(1+x).$$

Ans.

Que② Solve: $(2x-1)^2\frac{d^2y}{dx^2}+(2x-1)\frac{dy}{dx}-2y=8x^2-2x+3$

Sol.ⁿ The given eq.ⁿ is a legendre's linear eq.ⁿ.

Put $(2x-1)=e^t \Rightarrow t=\log(2x-1)$

$(2x-1)\frac{dy}{dx}=2Dy$; $(2x-1)^2\frac{d^2y}{dx^2}=4D(D-1)y$

where, $D=\frac{d}{dt}$

$\because 2x-1=e^t$
$x=\frac{e^t+1}{2}$

The eq.ⁿ becomes,

$$[4D(D-1)+2D-2]y=8\left(\frac{e^t+1}{2}\right)^2-2\left(\frac{e^t+1}{2}\right)+3$$

$$\Rightarrow (2D^2-D-1)y=e^{2t}+\frac{3}{2}e^t+2$$

Its AE is $2m^2-m-1=0 \Rightarrow m=1,-\frac{1}{2}$.

$\therefore CF=c_1e^t+c_2e^{-t/2}$ (Roots are real and distinct)

Now, $PI=\frac{1}{2D^2-D-1}\left(e^{2t}+\frac{3}{2}e^t+2\right)$

$$=\frac{1}{2D^2-D-1}e^{2t}+\left(\frac{3}{2}\right)\frac{1}{2D^2-D-1}e^t+(2)\frac{1}{2D^2-D-1}\cdot e^{0t}$$

$$=\frac{1}{8-3}e^{2t}+\frac{3}{2}t\cdot\frac{1}{4D-1}e^t+2\cdot\frac{1}{-1}$$

$$=\frac{e^{2t}}{5}+\frac{3}{2}t\left(\frac{e^t}{3}\right)+(-2)=\frac{e^{2t}}{5}+\frac{te^t}{2}-2.$$

The complete solution is

$$y=CF+PI=c_1e^t+c_2e^{-t/2}+\frac{e^{2t}}{5}+\frac{te^t}{2}-2.$$

$$=c_1(2x-1)+c_2(2x-1)^{-1/2}+\frac{(2x-1)^2}{5}+\frac{1}{2}(2x-1)\log(2x-1)-2 \quad \text{Ans.}$$

Que(1) Solve the simultaneous eq.^{n}:

$$\frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + x = \cos t$$

Sol.^{n} Taking $\frac{d}{dt} = D$, the above eq.^{n} becomes,

$$Dx + y = \sin t \quad \text{— (I)} \quad ; \quad Dy + x = \cos t \quad \text{— (II)}$$

Operating by D on eq.^{n} (II), we get $D^2y + Dx = D(\cos t) = -\sin t$ — (III)

Equating eq.^{ns} (I) and (III),

$$\begin{array}{r} Dx + y = \sin t \\ \underset{(-)}{D x} + \underset{(-)}{D^2 y} = \underset{(+)}{-}\sin t \\ \hline (1-D^2)y = 2\sin t \end{array} \Rightarrow (D^2-1)y = -2\sin t$$

Its AE is $m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$ (Real and Distinct)

$\therefore CF = c_1 e^t + c_2 e^{-t}$

$$PI = \frac{1}{D^2-1}(-2\sin t) = (-2)\frac{1}{-(1)^2-1}\sin t = (-2)\frac{1}{(-2)}\sin t$$

$PI = \sin t$

$\therefore \quad y = CF + PI = c_1 e^t + c_2 e^{-t} + \sin t$ — (IV)

from (II), $x = \cos t - Dy$

$$= \cos t - D(c_1 e^t + c_2 e^{-t} + \sin t)$$

$$= \cos t - c_1 e^t + c_2 e^{-t} + \cos t$$

$$x = -c_1 e^t + c_2 e^{-t} \quad \text{— (V)}$$

Eq.^{ns} (IV) and (V) are the solutions of given simultaneous eq.^{ns}

Que(2) Solve the following simultaneous equations:

$$\frac{dx}{dt} + 2x - 3y = 5t \; ; \; \frac{dy}{dt} - 3x + 2y = 2e^{2t}$$

Sol.^{n} Taking $\frac{d}{dt} = D$, the above eq.^{n} becomes,

$$Dx + 2x - 3y = 5t \Rightarrow (D+2)x - 3y = 5t \quad \text{— (I)}$$

Also, $Dy - 3x + 2y = 2e^{2t} \Rightarrow (D+2)y - 3x = 2e^{2t}$ — (II)

Multiplying eq. (I) by 3 and operating by $(D+2)$ on eq. (II),

$$3(D+2)x - 9y = 15t$$

$$(D+2)^2 y - 3(D+2)x = (D+2)2e^{2t} = 2\cdot(2e^{2t}) + 4e^{2t} = 8e^{2t}$$

$$\left[(D+2)^2 - 9\right] y = 15t + 8e^{2t}$$

$$\Rightarrow (D^2 + 4D - 5)y = 15t + 8e^{2t}$$

Its AE is $m^2 + 4m - 5 = 0$

$$m^2 + 5m - m - 5 = 0$$

$$m(m+5) - 1(m+5) = 0$$

$$(m+5)(m-1) = 0 \Rightarrow m = 1, -5$$

$\therefore$ $CF = c_1 e^{t} + c_2 e^{-5t}$ (Roots are real and distinct)

$$PI = \frac{1}{D^2+4D-5}\left(15t + 8e^{2t}\right) = 15 \cdot \frac{1}{D^2+4D-5} t + 8 \cdot \frac{1}{D^2+4D-5} e^{2t}$$

$$= 15 \frac{1}{-5+4D+D^2} t + 8 \frac{1}{4+8-5} e^{2t} = \frac{15}{(-5)} \cdot \frac{1}{\left(1 - \frac{4D}{5} - \frac{D^2}{5}\right)} \cdot t + \frac{8}{7} e^{2t}$$

$$= -3\left[1 - \left(\frac{4D}{5} + \frac{D^2}{5}\right)\right]^{-1} t + \frac{8}{7} e^{2t}$$

$$= -3\left[1 + \frac{4D}{5} + \frac{D^2}{5} + \cdots\right] t + \frac{8}{7} e^{2t}$$

$$= -3\left(t + \frac{4}{5}\right) + \frac{8}{7} e^{2t}$$

$$\therefore\ y = CF + PI$$
$$= c_1 e^{t} + c_2 e^{-5t} - 3\left(t + \frac{4}{5}\right) + \frac{8}{7} e^{2t}$$

from (ii),

$$(D+2)\left(c_1 e^{t} + c_2 e^{-5t} - 3t - \frac{12}{5} + \frac{8}{7} e^{2t}\right) - 3x = 2e^{2t}$$
$$\Rightarrow \cancel{c_1 e^{t} + c_2 e^{-5t}(-5) - 3 + \frac{8}{7} e^{2t}(2) - 3x = 2e^{2t}}$$
$$\Rightarrow \cancel{x = \frac{1}{3}\left(c_1 e^{t} - 5c_2 e^{-5t} - 3 + \frac{16}{7} e^{2t} - 2e^{2t}\right)}$$
$$\cancel{= \frac{1}{3}\left(c_1 e^{t} - 5c_2 e^{-5t} - 3 + \frac{2}{7} e^{2t}\right)}$$

Ans

from (ii),

$$(D+2)\left(c_1 e^{t} + c_2 e^{-5t} - 3t - \frac{12}{5} + \frac{8}{7} e^{2t}\right) - 3x = 2e^{2t}$$
$$\Rightarrow c_1 e^{t} + c_2 e^{-5t}(-5) - 3 - 0 + \frac{8}{7} e^{2t}(2) + 2c_1 e^{t}$$
$$+ 2c_2 e^{-5t} - 6t - \frac{24}{5} + \frac{16}{7} e^{2t} - 3x = 2e^{2t}$$
$$\Rightarrow c_1 e^{t} - 5c_2 e^{-5t} - 3 + \frac{16}{7} e^{2t} + 2c_1 e^{t} + 2c_2 e^{-5t}$$
$$- 6t - \frac{24}{5} + \frac{16}{7} e^{2t} - 3x = 2e^{2t}$$
$$\Rightarrow 3c_1 e^{t} - 3c_2 e^{-5t} - \frac{39}{5} + \frac{32}{7} e^{2t} - 6t - 2e^{2t} = 3x$$
$$\Rightarrow x = \frac{1}{3}\left[3c_1 e^{t} - 3c_2 e^{-5t} - \frac{39}{5} + \frac{18}{7} e^{2t} - 6t\right]$$
$$x = c_1 e^{t} - c_2 e^{-5t} - \frac{13}{5} + \frac{6}{7} e^{2t} - 2t$$

Ans

THANK YOU SO MUCH