Roll No. ....

D–3751

M. A./M. Sc. (Previous) EXAMINATION, 2020

MATHEMATICS

Paper First

(Advanced Abstract Algebra)

Time : Three Hours] [*Maximum Marks : 100*

Note : Attempt any *two* parts from each question. All questions carry equal marks.

Unit—I

1. (a) Let G be a group. Prove that if G is solvable, then every subgroup of G and homomorphic image of G are solvable. Conversely, if N is normal subgroup of G such that N and G/N are solvable then G is solvable.

   (b) Find the splitting field of $f(x) = x^4 - 2 \in Q[x]$ over Q and its degree of extension.

   (c) Let E be an algebraic extension of a field F containing an algebraic closure $\overline{F}$ of F. Then show that the following are equivalent :

   (i) Every irreducible polynomial in F [x] that has a root in E splits into linear factors in E.

   (ii) E is the splitting field of a family of polynomial in F [x].

   (iii) Every embedding σ of E in $\overline{F}$ that keeps each element of F fixed maps E onto E.

Unit—II

2. (a) Suppose that the field F has all nth root of unity and suppose that $a \neq 0$ is in F. Let $x^n - a \in F[x]$ and let K be its splitting field over F. Then show that :

   (i) K = F (u) where u is any root of $x^n - a$.

   (ii) The Galois group of $x^n - a$ over F is abelian.

   (b) If splitting field of the polynomial $x^4 - 3x^2 + 4$ over Q is K, then find the Galois group of K over Q.

   (c) Let E be a finite separable extension of a field F. Then show that the following are equivalent :

   (i) E is a normal extension of F.

   (ii) F is the fixed field of G (E/F)

   (iii) [E : F] = | G (E/F) |

Unit—III

3. (a) State and prove Hilbert basis theorem.

   (b) Show that ring :

$$R = \begin{pmatrix} Z & Q \\ 0 & Q \end{pmatrix}$$

   is right noetherian.

   (c) Let M be a finitely generated free module over a commutative ring R. Then show that all bases of M have the same number of elements.

Unit—IV

4. (a) Let V be a vector space of polynomials of degree ≤ 3, and let $T : V \to V$ be a linear transformation defined by :

$$T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \alpha_0 + \alpha_1(x+1) + \alpha_2(x+1)^2 + \alpha_3(x+1)^3$$

Compute the matrix of T relative to bases :

(i) $(1, x, x^2, x^3)$

(ii) $(1, 1+x, 1+x^2, 1+x^3)$

Denote the above matrices by A and B respectively. Find a matrix C such that $B = CAC^{-1}$.

(b) Let U and V be two vector spaces over a field F, of dimensions m and n respectively. Then show that Hom (U, V) is a vector space over F of dimension mn.

(c) Let $T \in A(V)$ and $m(x)$ be the minimal polynomial of T over F. Show that for $0 \neq v \in V$.

(i) $F[T]v = \{[f(T)]v \mid f(T) \in F[T]\}$ is a non-zero subspace of V containing v.

(ii) There exists a unique non-zero monic polynomial $m_v(x)$ over F such that :

(1) $[m_v(T)]v = 0$

(2) For any $f(x) \in F[x]$, $[f(T)]v = 0 \Rightarrow m_v(x) \mid f(x)$

(3) $m_v(x) \mid m(x)$

(4) deg $m_v(x) = \dim_F F[T]_v$

Unit—V

5. (a) Let R be a principal ideal domain and let M be any finitely generated R-module. Then show that :

$$M \simeq R^S \oplus R/Ra_1 \oplus \ldots.. \oplus R/Ra_r$$

a direct sum of cyclic modules, where the a_i are non-zero non-units and $a_i \mid a_{i+1}, i = 1, \ldots.., r-1$.

(b) Find invariant factors, elementary divisors and the Jordan canonical form of the matrix :

$$\begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}$$

(c) Let $T \in \text{Hom}_F(V, V)$ and let $f_1(x), \ldots.., f_n(x)$ be the invariant factor of $A - xI$, where A is a matrix of T. Then show that :

$$V \simeq \frac{F[x]}{(f_1(x))} \oplus \ldots. \oplus \frac{F[x]}{(f_n(x))}.$$