Roll No. ....

E–309

M. A./M. Sc. (First Semester)

EXAMINATION, Dec.-Jan., 2020-21

MATHEMATICS

Paper First

(Advanced Abstract Algebra—I)

Time : Three Hours] [*Maximum Marks : 80*

[*Minimum Pass Marks : 16*

Note : Attempt all Sections as directed.

Section—A 1 each

(Objective/Multiple Choice Questions)

Note : Attempt all questions.

Choose the correct answer :

1. S_n is solvable :

   (a) for each positive integer *n*

   (b) for each prime number *n*

   (c) for $n \leq 4$

   (d) for $n \geq 5$

P. T. O.

2. Which one of the following is correct for solvable group ?
   (a) Abelian group is not solvable
   (b) Nilpotent group is not solvable
   (c) Cyclic group is not solvable
   (d) p-group is solvable

3. Jordan-Holder theorem implies :
   (a) fundamental theorem of algebra
   (b) fundamental theorem of arithmetic
   (c) fundamental theorem of homomorphism
   (d) fundamental theorem of field

4. If H is the maximal normal subgroup of G, then :
   (a) G/H is abelian
   (b) G/H is cyclic
   (c) G/H is simple
   (d) G/H is p-group

5. Which order of the group is not necessarily nilpotent ?
   (a) 27
   (b) 24
   (c) 32
   (d) 25

6. Set of all algebraic elements of K over a field F is :
   (a) Ring
   (b) Integral domain
   (c) Division ring
   (d) Field

7. Degree of splitting field of $x^p - 1$ over Q, p being a prime number is :

(a) p

(b) $p - 1$

(c) $2p$

(d) $p!$

8. Which one of the following statements is not true ?

(a) C is algebraic extension of R.

(b) C is normal extension of R.

(c) C is separable extension of R.

(d) C is infinite extension of R.

9. There exists no field of order :

(a) 18

(b) 81

(c) 125

(d) 72

10. Which one of the following is algebraically closed field ?

(a) Q

(b) R

(c) C

(d) $Q(\sqrt{2})$

11. Characteristic of a finite field is :

(a) any positive integer

(b) prime number

(c) in the form p^n, p is a prime

(d) n^P, p is a prime number

12. A field F is called perfect field if :

    (a) all finite extensions of F are normal.

    (b) all finite extensions of F are separable.

    (c) all finite extensions of F are inseparable.

    (d) all finite extensions of F are simple extension

13. Which is conjugate of w over Q ?

    (a) i

    (b) π

    (c) w^2

    (d) e

14. Let G be a finite group of automorphism of a field K, F_0 is the fixed field under G, then $[K, F_0] = o(G)$ is called :

    (a) Galois's theorem

    (b) Kronecker's theorem

    (c) Artin's theorem

    (d) Langrage's theorem

15. Which is the fixed field of $Q(\sqrt{2})$ under Aut $Q(\sqrt{2})$?

    (a) Q

    (b) $Q(\sqrt{2})$

    (c) R

    (d) C

16. Which one is incorrect :
   (a) All polynomials of degree 2 are solvable by radicals.
   (b) All polynomials of degree 3 are solvable by radicals.
   (c) All polynomials of degree 4 are solvable by radicals.
   (d) All polynomials of degree 5 are solvable by radicals.

17. Which one of the following is not elementary symmetric function of x_1, x_2, x_3 :
   (a) $x_1 + x_2 + x_3$
   (b) $x_1 x_2 x_3$
   (c) $x_1 x_2 + x_2 x_3 + x_3 x_1$
   (d) $x_1^2 + x_2^2 + x_3^2$

18. The generic polynomial $p_n(t)$ of degree *n* is not solvable by radicals over $F(a_1, a_2, \ldots\ldots, a_n)$ where $a_1, a_2, \ldots\ldots\ldots, a_n$ are elementary symmetric function :
   (a) for $n \geq 2$
   (b) for $n \geq 3$
   (c) for $n \geq 4$
   (d) for $n \geq 5$

19. The roots of polynomial $x^n - 1$ over Q form :
   (a) A group but not abelian
   (b) Abelian group but not cyclic
   (c) Cyclic group
   (d) None of the above

20. Let $F \leq E \leq K$, then which one of the following is correct ?

(a) G (K, F) $\leq$ G (K, E)

(b) G (K, E) $\leq$ G (K, F)

(c) G (K, E) = G (K, F)

(d) o (G (E, F)) = o (G (K, F))

Section—B 2 each

(Very Short Answer Type Questions)

Note : Attempt all questions.

1. Define nth derived set of a group.
2. Define simple extension.
3. What is the degree of $Q(\sqrt{2}, i)$ over Q ?
4. Define algebraically closed field.
5. Define primitive element.
6. Write splitting field of $x^3 - 2$ over Q.
7. Define fixed field of group of automorphism G of R.
8. State Abel's theorem.

Section—C 3 each

(Short Answer Type Questions)

Note : Attempt all questions.

1. Show that every group of order p^n is nilpotent.
2. Show that A_n is not solvable for $n \geq 5$.
3. Show that every finite extension is algebraic extension but converse need not be true.
4. Show that an irreducible polynomial $f(x)$ over a field F of characteristic $p > 0$ is inseparable if and only if $f(x) \in F[x^p]$.

5. Show that for energy prime number p and $n \in N$ there exists a field having p^n elements.

6. Let E and K be any two fields. If $\sigma_1, \sigma_2, \ldots\ldots\ldots, \sigma_n$ are n distinct monomorphism of E into K then show that they are linearly independent.

7. Show that multiplicative group of any finite field is cyclic.

8. Prove that $x^5 - 4x + 2$, is not solvable by radicals over Q.

Section—D 5 each

(Long Answer Type Questions)

Note : Attempt all questions.

1. Prove that a group G is nilpotent if and only if G has a normal series :

$$(\rho) \leq G_0 \leq G_1 \leq \ldots\ldots\ldots \leq G_m = G$$

such that $\frac{G_i}{G_i - 1} \leq z\left(\frac{G}{G_{i-1}}\right)$ for all i = 1, 2 .... m.

Or

Show that every subgroup and homomorphic image of nilpotent group is nilpotent.

2. Show that an element $a \in K$ is algebraic over F if and only if $[F(a) : F]$ is finite.

Or

Show that if L is algebraic extension of K and K is algebraic extension of F, then L is also algebraic extension of F.

3. Let F be a field having q elements. Then prove that F is the splitting field of $x^9 - x$ over prime subfield.

Or

Let E be an algebraic extension of field F and let $\sigma : F \to L$ be an embedding of F into an algebrically closed field L. Then show that σ can be extended to an embedding $\eta : E \to L$.

4. Let K be a finite normal extension of field F of characteristic zero. E be a subfield of K containing F. Then prove that :

(i) $[K : E] = o\ G(K, E)$ and

$[E : F] =$ index of $G(K, E)$ in $G(K, F)$

(ii) E is normal extension of F if and only if G (K, E) is normal subgroup of G (K, F).

Or

Prove that $f(x) \in F[x]$ is solvable by radicals over F if and only if its splitting field E over F has solvable Galois group G (E, F).