Roll No. ....

E–312

M. A./M. Sc. (First Semester)

EXAMINATION, Dec.-Jan., 2020-21

MATHEMATICS

Paper Fourth

(Advanced Complex Analysis—I)

Time : Three Hours] [*Maximum Marks : 80*

[*Minimum Pass Marks : 16*

Note : Attempt all Sections as directed.

Section—A 1 each

(Objective/Multiple Choice Questions)

Note : Attempt all questions.

Choose the correct answer :

1. $\int_L |dz|$, where L is any rectifiable arc joining the points $z = a$ and $z = b$ is equal to :

   (a) $|b - a|$

(b) $b - a$

(c) arc length of L

(d) 0

2. The value of $\oint_{|z|=3} \frac{e^z}{z-2} dz$ is :

(a) 0

(b) $2\pi i e^3$

(c) $2\pi i e^2$

(d) $2\pi i$

3. The number of zeros of the function $f(x) = \sin\left(\frac{1}{z}\right)$ is :

(a) 3

(b) 4

(c) infinite

(d) no zeros exist

4. If $f(z) = \frac{z}{z^2 - 1}$, then at $z = 1$, $f(z)$ has a pole of order :

(a) One

(b) Two

(c) Zero

(d) None of these

5. If a polynomial is of degree *n*, then the number of zeros it has :

(a) One

(b) *n*

(c) No zeros

(d) None of these

6. If $f(z)$ be analytic inside and on a closed contour C and let $f(z) \neq 0$ inside C, then $|f(z)|$ attains its mininum value :

(a) on C and not inside C

(b) inside C and not on C

(c) inside and on C

(d) None of these

7. A function $f(z)$ which is analytic in every finite region of the *z*-plane is called :

(a) an entire function

(b) a meromorphic function

(c) integral function

(d) Both (a) and (b) are correct.

8. The series $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$ for $|z| < \infty$ represents the following function :

(a) $\sin z$

(b) $\cos z$

(c) $\tan z$

(d) None of these

9. If $z = 0$ is a simple pole of $f(z)$, then the residue at this pole is given by :

(a) $\lim_{z \to a} (z - a) f(z)$

(b) $\lim_{z \to 0} (z - a) f(z)$

(c) $\lim_{z \to 0} = f(z)$

(d) $f(a)$

10. The residue of $f(z) = \frac{z}{(z-1)(z-2)}$ at $z = 1$ is :

(a) 0

(b) 1

(c) – 1

(d) ∞

11. The residue of the function $\dfrac{1}{\left(z^2+1\right)^3}$ at $z = i$ is :

(a) $\dfrac{3}{16i}$

(b) $\dfrac{3i}{16}$

(c) $\dfrac{-3i}{16i}$

(d) None of these

12. If $f(z)$ is a conformal mapping, then :

(a) It preserves the magnitude of angle but not sense

(b) It preserves the sense of angle but not magnitude

(c) It preserves both magnitude and sense of angles

(d) It preserves neither the magnitude nor the sense of angles

13. The transformation $w = \dfrac{az+b}{cz+d}$ is said to be normalized if the value of *ad*-*bc* is :

(a) 1

(b) 0

(c) ∞

(d) None of these

14. The equation $\alpha z + \bar{\alpha}\,\bar{z} = c$, where c is a real number represents a :

(a) circle

(b) straight line

(c) Both (a) and (b)

(d) None of these

15. $C \quad G, \Omega$ is a :

(a) Metric space

(b) Complete metric space

(c) Vector space

(d) None of these

16. If f is analytic in a domain D and not constant, then $w = f \quad z$ maps open sets of D onto :

(a) Closed sets in w-plane

(b) Open sets in w-plane

(c) Both (a) and (b)

(d) None of these

17. The inverse of the point z with respect to the circle $|z| = r$ is :

(a) $\frac{r^2}{z}$

(b) $\frac{r}{\overline{z}}$

(c) $\frac{r}{z}$

(d) $\frac{r^2}{\overline{z}}$

18. Fixed points of bilinear transformation $w = \frac{z}{2 - z}$ are :

(a) 0, 1

(b) 1, 2

(c) 0, 2

(d) 1, 3

19. The inverse transformation of $w = \frac{z+1}{z+3}$ is :

(a) $z = \frac{1 - 3w}{w - 1}$

(b) $\frac{3w - 1}{w - 1}$

(c) $\frac{1 - 3w}{w + 1}$

(d) $\frac{1 + 3w}{w - 1}$

20. Under the transformation of $w = z + 2 - i$, the line $x = 1$ is transformed into the line :

(a) $v = 1$

(b) $u = -1$

(c) $v = 2$

(d) $u = 3$

Section—B 2 each

(Very Short Answer Type Questions)

Note : Attempt all questions.

1. Evaluate $\int_C \frac{z^2 - z + 1}{z - 1}\, dz$, where C is the circle $|z| = \frac{1}{2}$.

2. State the Cauchy Integral formula for the derivative of an analytic function.

3. Define zero of an analytic function with an example.

4. State the fundamental theorem of Algebra.

5. Define the pole of an analytic function.

6. Find the residue of $\frac{1}{{z^2 + a^2}^{2}}$ at $z = ia$.

7. Define Bilinear transformation.

8. State the Montel's theorem.

Section—C 3 each

(Short Answer Type Questions)

Note : Attempt all questions.

1. Expand $\dfrac{z^2 - 1}{z + 2 \quad z + 3}$ in a Laurent's series valid for the region $2 < |z| < 3$.

2. Find the kind of the singularity of the function $\sin\left(\dfrac{1}{1 - z}\right)$ at point $z = 1$.

3. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

4. Apply calculus of residues to prove that :

$$\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{5 + 4\cos\theta} = \frac{\pi}{6}$$

5. Consider the transformation $w = e^{i\pi/4} z$ and determine the region in the w-plane corresponding to the triangular region bounded by the lines $x = 0, y = 0$ and $x + y = 1$ in the z-plane.

6. Show that the transformation $w = \dfrac{2z + 3}{z - 4}$, maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$.

7. If $F \subset C(G, \Omega)$ is equicontinuous at each point of G, then F is equicontinuous over each compact subset of G, prove it.

8. Define Normality and Equicontinuity.

Section—D 5 each

(Long Answer Type Questions)

Note : Attempt all questions.

1. State and prove Poisson's integral formula for a circle.

Or

State and prove Rouche's theorem.

2. By the method of contour integration, prove that :

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

Or

By contour integration, show that :

$$\int_0^\infty \frac{\sin x}{x(x^2+a^2)}\,dx = \frac{\pi}{2a^2}\left(1-e^{-a}\right), \quad a > 0$$

3. Show that every bilinear transformation maps circles or straight lines into circles or straight line.

Or

State and prove Liouville's theorem.

4. State and prove Hurwitz's theorem for the spaces of analytic functions.

Or

If f is analytic in a domain D and not constant, then show that $w = f(z)$ maps open sets of D onto the open sets in the w-plane.