E-519

M.A./M.Sc. (Second Semester)(Main/ATKT)
EXAMINATION, June 2021
MATHEMATICS
Paper First
(Advanced Abstract Algebra-II)

Time: Three Hours] [*Maximum Marks* 80

Note: Attempt all sections as directed.

Section-A

1 each

(Obejctive/Multiple Choice Questions)

Note: Attempt all questions.

Choose one correct answer out of four alternative answers (a) through (d).

1. If A and B are R–submodules of R–module M and N, respectively, then

   (a) $\frac{M \times N}{A \times B} \simeq \frac{A}{M} \times \frac{N}{B}$

   (b) $\frac{A \times B}{M \times N} \simeq \frac{A}{M} \times \frac{B}{M}$

   (c) $\frac{M \times N}{A \times B} \simeq \frac{A}{M} \times \frac{B}{N}$

   (d) $\frac{M \times N}{A \times B} \simeq \frac{M}{A} \times \frac{N}{B}$

2. If R be a Euclidean ring, then any finitely generated R–module M is the direct sum of a finite number of :

   (a) submodules

   (b) simple module

   (c) cyclic submodules

   (d) none of the above

3. A Boolean noetherian ring is finite and is a finite direct product of fields with ______ element(s).

   (a) one

   (b) two

   (c) three

   (d) four

4. If R is noetherian, then each ideal contains a finite product of ______ ideals.

   (a) nil

   (b) nilpotent

   (c) prime

   (d) none of the above

5. A ring A is said to be an algebra over F, if A is a vector space over F such that

   (a) $\alpha(ab) \neq (\alpha a)b = a(\alpha b)$

   (b) $\alpha(ab) = (\alpha a)b \neq a(\alpha b)$

   (c) $\alpha(ab) \neq (\alpha a)b \neq a(\alpha b)$

   (d) $\alpha(ab) = (\alpha a)b = a(\alpha b)$

6. Let A be an algebra, with unit element over F, and let dimension of A over F be m. Then every element in A satisfies some non trivial polynomial in $F[x]$ of degree

(a) at least m
(b) at most m
(c) m
(d) none of the above

7. If V is finite dimensional vector space over F, then $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is :

(a) 0
(b) 1
(c) any nonzero value
(d) none of the above

8. If $T : \mathbb{R}^2 \to \mathbb{R}^2$, then which of the following is a linear transformation?

(a) $T(x, y) = (|x|, y)$
(b) $T(x, y) = (x + 1, y - x)$
(c) $T(x, y) = (2x + 3y, 5x - 6y)$
(d) none of the above

9. Let $A : \mathbb{R}^6 \to \mathbb{R}^5$ and $B : \mathbb{R}^6 \to \mathbb{R}^7$ be two linear transformations, then which of the following is not possible? :

(a) A and B both are onto
(b) A and B are one-one
(c) A is one-one and B is not one-one
(d) A is onto and B is one-one

10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (2x + 3y, 4x - 5y)$, then the matrix representation $[T]_S$ of T relative to the basis $S = \{u_1, u_2\} = \{(1, -2), (2, -5)\}$ is :

(a) $\begin{pmatrix} 8 & -6 \\ -11 & 11 \end{pmatrix}$
(b) $\begin{pmatrix} -6 & 8 \\ -11 & 11 \end{pmatrix}$
(c) $\begin{pmatrix} -6 & -11 \\ 8 & 11 \end{pmatrix}$
(d) $\begin{pmatrix} 8 & 11 \\ -6 & -11 \end{pmatrix}$

11. Let $T \in A(V)$ be nilpotent and let the subspace M of V of dimension m be cyclic with respect to T, then :

(a) $\dim mT^k = m - k \quad \forall k \geq m$
(b) $\dim mT^k = m - k \quad \forall k \leq m$
(c) $\dim mT^k = k - m \quad \forall k \geq m$
(d) $\dim mT^k = m + k \quad \forall k \leq m$

12. Let M be an $R-$module and let N and P be submodules of M with $P \subseteq N$. Then $\frac{M}{N} \simeq \frac{M}{P} / \frac{N}{P}$ is known as :

(a) First isomorphism theorem
(b) Second isomorphism theorem
(c) Third isomorphism theorem
(d) none of the above

13. Which of the following statement is not true?

   (a) An ideal P in a commutative ring R is prime if $P = R$ and P is such that $ab \in P$ then $a \in P$ or $b \in P$.

   (b) An ideal P in a commutative ring R is prime if $P \neq R$ and P is such that $ab \in P$ then $a \in P$ or $b \in P$.

   (c) If F is a field, then $F[x]$ is a PID.

   (d) An $R-$module M is a torsion free module if $\text{Tor}M = 0$.

14. Let R be a PID, and let F be a free $R-$module with a basis consisting of n elements. Then any submodule K of F is also free with a basis consisting of m elements, such that

   (a) $m = n$

   (b) $m \geq n$

   (c) $m \leq n$

   (d) none of the above

15. Let $A = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}$ where, $a \neq 0$. If $m(x)$ and $\Delta(x)$ are the minimal and the characteristic polynomial respectively of A, then :

   (a) $m(x) \neq \Delta(x)$

   (b) The degree of the minimal polynomial is 1

   (c) $m(x) = \Delta(x)$

   (d) A has two distinct characteristic roots.

16. Let $A = \begin{pmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}$ be the Jordan block of order 4, then :

   (a) Characteristic polynomial is $(t-7)^4$

   (b) minimal polynomial is $(t-7)^4$

   (c) 7 is the only eigenvalue

   (d) All of the above

17. The abelian group generated by x_1 and x_2 subject to $2x_1 = 0, 3x_2 = 0$ is isomorphic to :

   (a) $\mathbb{Z}$

   (b) $\mathbb{Z}_2$

   (c) $\mathbb{Z}_6$

   (d) none of the above

18. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear mapping satisfying $T(e_1) = e_2, T(e_2) = e_3, T(e_3) = 0, T(e_4) = e_3$, where e_1, e_2, e_3, e_4 is the standard basis of $\mathbb{R}^4$. Then :

    (a) T is idempotent
    (b) T is invertible
    (c) Rank $T = 3$
    (d) T is nilpotent

19. Let V be a vector space and T be a linear operator on V. If W is a subspace of V, then W is invariant under T, if

    (a) $T(W) \subseteq W$
    (b) $W \subseteq T(W)$
    (c) $T(W) = W$
    (d) none of the above

20. The rational canonical form of a 2×2 matrix A with invariant factor $(x-3)(x-1)$ is :

    (a) $\begin{pmatrix} 0 & -3 \\ 1 & 4 \end{pmatrix}$
    (b) $\begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}$
    (c) $\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$
    (d) none of the above

Section-B

2 each.

(Very Short Answer Type Questions)

Note: Attempt all questions. Answer in 2-3 sentences.

1. Define companion matrix.
2. Define Nilpotent transformation.
3. State Noether-Lasker theorem.
4. Define semisimple module.
5. State primary Decomposition theorem.
6. Show that $\alpha \in F$ is an eigen value of $T \in A(V)$ then $T - \alpha I$ is singular.
7. Define rank of a linear transformation.
8. Write down companion matrix of $f(x) = x^3 - 4x^2 + 5x + 9$.

Section-C

3 each.

(Short Answer Type Questions)

Note: Attempt all questions.

1. Show that the kernel of a module homomorphism is a submodule.

2. Find all Jordan normal form for 6×6 matrix having t^3 as the minimal polynomial.

3. If $0 \neq T \in A(V)$ and $S \in A(V)$ are invertible, then show that T and $S^{-1}TS$ have same minimal polynomials.

4. Let R be a PID, and let M be an R-module, then show that $\text{Tor}M = \{x \in M | x \text{ is torsion}\}$ is a submodule of M.

5. Let A be a 4×4 matrix with minimal polynomial $m(t) = (t^2+1)(t^2-3)$. Then find the rational canonical form of A, if A is the matrix over $\mathbb{R}$ and $\mathbb{C}$.

6. Let A be a $m \times n$ matrix over R. If $M_{ij}(\alpha) = 1 + \alpha e_{ij}$, then show that $M_{ij}(\alpha)A$ is the matrix obtained from A by multiplying the j^{th} row by α and adding it to the i^{th} row. Also $M_{ij}^{-1}(\alpha) = 1 - \alpha e_{ij}(i \neq j)$.

7. Let the linear transformation $T \in A(V)$ be nilpotent, if $\alpha_0 \neq 0$ then show that $\alpha_0 + \alpha_1 T + \cdots + \alpha_m T^m$, is invertible where $\alpha_i \in F, 0 \leq i \leq m$.

8. Show that every submodule and every quotient module of an artinian module is artinian.

Section-D

5 each.

(Long Answer Type Questions)

Note: Attempt all questions.

1. State and prove Hilbert basis theorem.

OR

If M is a free module with a basis $(e_1, e_2, \cdots, e_n)$, then prove that $M \simeq \mathbb{R}^n$.

2. Let U and V be two vector spaces over a field F of dimensions m and n respectively. Then show that $\text{Hom}(U, V)$ is a vector space over F of dimension mn.

OR

Let V be finite dimensional vector space over field F and let $T(\neq 0) \in A(V)$. Then prove that V has a vector v such that the minimal polynomials of T and of v relative to T are equal.

3. Find the Jordan canonical form of

$$A = \begin{pmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

OR

State and prove Fundamental structure theorem for finitely generated modules over a PID.

4. Find the invariant factors of the following matrix over $\mathbb{Q}[x]$:

$$\begin{pmatrix} 5-x & 1 & -2 & 4 \\ 0 & 5-x & 2 & 2 \\ 0 & 0 & 5-x & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

OR

Let W be a subspace of V and let $T \in Hom(V,V)$ such that $TW \subseteq W$. Show that W is a T--cyclic subspace if and only if there exists an element $w \in W$ such that $w, Tw, \cdots, T^{k-1}w$ is a basis of W for some $k \geq 1$.

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