Roll No. ....

E–520

M. A./M. Sc. (Second Semester) (Main/ATKT) EXAMINATION, May-June, 2021

MATHEMATICS

Paper Second

(Real Analysis—II)

Time : Three Hours] [*Maximum Marks : 80*

Note : Attempt all Sections as directed.

Section—A 1 each

(Objective/Multiple Choice Questions)

Note : Attempt all questions.

Choose the correct answer :

1. Let $f:[a,b]\to R$ be bounded function and α be monotonic increasing function. If P* is a refinement of the partition P of the interval [*a*, *b*], then :
   (a) $U(P, f, \alpha) \le U(P^*, f, \alpha)$
   (b) $U(P, f, \alpha) \le L(P, f, \alpha)$
   (c) $L(P, f, \alpha) \le L(P^*, f, \alpha)$
   (d) None of the above

2. The value of $\int_0^1 x\, d[x]$ is :
   (a) 0
   (b) 1
   (c) $\frac{1}{2}$
   (d) –1
3. If RS(P, Q, *f*, α) be RS-sum of *f* relative to α on [*a*, *b*] and corresponding to the partition P and the intermediate partition Q, then :
   (a) $U(P, f, \alpha) \le RS(P, Q, f, \alpha)$
   (b) $RS(P, Q, f, \alpha) \le U(P, f, \alpha)$
   (c) $L(P, f, \alpha) \le RS(P, Q, f, \alpha)$
   (d) Both (b) and (c)
4. If $f \in RS(\alpha)$ on [*a*, *b*], then which of the following is true ?
   (a) $4f \in RS(\alpha)$
   (b) $|f| \in RS(\alpha)$
   (c) $f^2 \in RS(\alpha)$
   (d) All of the above
5. Which of the following is not true ?
   (a) $m^*\phi = 0$
   (b) $m^*(\{x\}) = 0$
   (c) $A \subseteq B \to m^*(B) \le m^*A$
   (d) $m^*(A + x) = m^*(A)$

6. If $m^*(\mathrm{E}) = 0$, then E :
   (a) is measurable
   (b) is never measurable
   (c) may not be measurable
   (d) None of the above

7. Which of the following is true ?
   (a) $(-\infty, a]$ is measurable.
   (b) (a, ∞) is measurable.
   (c) Both (a) and (b)
   (d) None of the above

8. Let f be a function defined on a measurable set E. Then f is measurable iff $f^{-1}(\mathrm{G})$ is measurable :
   (a) For any open set G in R
   (b) For any closed set G in R
   (c) Both (a) and (b)
   (d) None of the above

9. Which of the following is not true ?
   (a) A simple function is always measurable.
   (b) Every step function is a simple function.
   (c) A continuous function on measurable set is measurable.
   (d) Every measurable function is continuous

10. A function $f : \mathrm{R} \to \{0, 1\}$ defined by :

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

   (a) is measurable.
   (b) is continuous.
   (c) is measurable but not continuous.
   (d) is continuous but not measurable.

11. A function $f : [0, 1] \to \mathrm{R}$ defined by :

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

   (a) is Lebesgue integrable.
   (b) is Riemann integrable.
   (c) is Lebesgue integrable but not Riemann integrable.
   (d) is Riemann integrable but not Lebesgue integrable.

12. Consider the following statements :
   (I) If $f = 0$ a. e. on E then $\int_{\mathrm{E}} f = 0$.
   (II) If $\int_{\mathrm{E}} f = 0$ then $f = 0$ a. e. on E.
   (a) (I) is true and (II) is false.
   (b) (II) is true and (I) is false.
   (c) (I) and (II) both are true.
   (d) (I) and (II) both are false.

13. If f is an integrable function, then $\left|\int f\right| = \int |f|$ when :
   (a) $f \ge 0$ a. e.
   (b) $f \le 0$ a. e.
   (c) Either $f \ge 0$ a. e. or $f \le 0$ a. e.
   (d) None of the above

14. Consider the following statements :

(I) A bounded monotone function is a function of bounded variation.

(II) A continuous function is of bounded variation.

(a) (I) is true and (II) is false.

(b) (II) is true and (I) is false.

(c) (I) and (II) both are true.

(d) (I) and (II) both are false.

15. Consider the following statements :

(I) Absolutely continuous function on $[a, b]$ is of bounded variation.

(II) If $f' = 0$ a. e., then f is constant function.

(a) (I) and (II) both are true

(b) (I) is true and (II) is false.

(c) (II) is true and (I) is false.

(d) (I) and (II) both are false.

16. Let $f(x) = |x|$, then $D^+ f(0) =$

(a) –1

(b) 0

(c) 1

(d) None of the above

17. Let X = [0, 16] and $f : X \to R$ be a function defined by $f(x) = x^{-1/4}$, $x \in X$. Then :

(a) $f \in L^4(X)$

(b) $f \in L^1(X)$

(c) Both (a) and (b)

(d) None of the above

18. If $f, g \in L^2[a, b]$, then :

(a) $f.g \in L^2$

(b) $f + g \in L^1$

(c) $f.g \in L^1$

(d) None of the above

19. If $0 < p < 1$ and $f, g \in L^p(\mu)$ be non-negative, then :

(a) $\| f + g \|_p \le \| f \|_p + \| g \|_p$

(b) $\| f + g \|_p \ge \| f \|_p + \| g \|_p$

(c) $\| fg \| \le \| f \|_p \| g \|_p$

(d) None of the above

20. Which of the following is a pair of conjugate numbers ?

(a) 3, –3

(b) 3, 3

(c) 2, 2

(d) 4, 4

Section—B $1\frac{1}{2}$ each

(Very Short Answer Type Questions)

Note : Attempt all questions.

1. Define Riemann-Stieltjes sum.
2. Define Rectifiable curves.
3. Define Lebesgue measurable set.
4. Define simple function.
5. Define Lebesgue integral of non-negative measurable function.
6. Write the statement of Lebesgue Differentiation theorem.
7. Write the statement of Jordan Decomposition theorem.

8. Write the statement of Lebesgue Monotone Convergence theorem.
9. Define L^p space.
10. Write the statement of Schwarz's inequality.

Section—C $2\frac{1}{2}$ each

(Short Answer Type Questions)

Note : Attempt all questions.

1. If $f, g \in RS(\alpha)$, then show that $f.g \in RS(\alpha)$.
2. Let $f \in RS(\alpha)$ on [*a*, *b*] and *m*, M are the bounds of the function *f*. Then show that :

$$m[\alpha(b)-\alpha(a)] \le \int_a^b f\, d\alpha \le M[\alpha(b)-\alpha(a)].$$

3. If $m^*(E) = 0,$ then show that E is measurable.
4. Let *f* be a measurable function defined on a measurable set E. Then prove that $|f|$ is also measurable on E.
5. Evaluate the Lebesgue integral of the function $f : [0,1] \to R$ defined by :

$$f(x) = \begin{cases} \dfrac{1}{x^{3/2}}, & \text{if } 0 < x \le 1 \\ 0 \ , & \text{if } x = 0 \end{cases}$$

and show that *f* is Lebesgue integrable on [0, 1].

6. Prove that difference of two measurable sets is a measurable set.
7. If the function *f* assumes its maximum at *c*, then prove that $D^+ f(c) \le 0$ and $D- f(c) \ge 0$.
8. If *f* and *g* are absolutely continuous functions on [*a*, *b*], then prove that *f* + *g* is also absolutely continuous.
9. If $f \in L^p[a,b]$ and $g \le f$, then prove that $g \in L^p[a,b]$.
10. If $f, g \in L^2$, then show that :

$$\| f+g \|_2 \le \| f \|_2 + \| g \|_2$$

P. T. O.

Section—D 4 each

(Long Answer Type Questions)

Note : Attempt all questions.

1. Let *f* be a bounded function and α be a monotonically increasing function on [*a*, *b*]. Then show that $f \in RS(\alpha)$ on [*a*, *b*] iff for every $\in > 0$, there exists a partition P such that :

$$U(P, f, \alpha) - L(P, f, \alpha) < \in .$$

Or

Let γ be a continuously differentiable curve on [*a*, *b*]. Then show that γ is rectifiable and $\wedge_\gamma(a,b) = \int_a^b |\gamma'(t)|\, dt$.

2. Show that a countable union of measurable sets is a measurable set.

Or

Prove that every Borel set in R is measurable.

3. State and prove Fatou's lemma.

Or

Let *f* be a bounded function defined on [*a*, *b*]. If *f* is Riemann integrable on [*a*, *b*], then show that it is Lebesgue integrable on [*a*, *b*] and :

$$R\int_a^b f d\alpha = \int_a^b f d\alpha .$$

4. Let *f* be a Lebesgue integrable function on [*a*, *b*]. Then show that the indefinite integral of *f* is a continuous function of bounded variation on [*a*, *b*].

Or

If *f* is absolutely continuous on [*a*, *b*] and $f' = 0$ a. e., then show that *f* is constant function.

5. State and prove Hölder's inequality for L^p spaces.

Or

State and prove Minkowski's inequality for L^p spaces.