Roll No. ....

E–998

M. A./M. Sc. (Fourth Semester) (Main/ATKT)

EXAMINATION, May-June, 2021

MATHEMATICS

Paper Fourth (B)

(Wavelets—II)

Time : Three Hours] [*Maximum Marks : 80*

Note : Attempt all Sections as directed.

Section—A 1 each

(Objective/Multiple Choice Questions)

Note : Attempt all questions.

Choose the correct answer :

1. If f and g are in $L^2(\mathbf{R})$, then $\int_{-\infty}^{\infty} f'(x)\, g(x)\, dx =$

(a) $\int_0^1 f'(x)\, g'(x)\, dx$

(b) $\int_{-\infty}^{\infty} f(x) g'(x) \, dx$

(c) $-\int_{-\infty}^{\infty} f(x) g'(x) \, dx$

(d) None of the above

2. Let H be a Hilbert space and $\{e_j : j = 1, 2, ...\}$ be the elements of H. Then $f = \sum_{j=1}^{\infty} \langle f, e_j \rangle e_j$ implies :

(a) $\| f \|^2 \leq \sum_{j=1}^{\infty} \left| \langle f, e_j \rangle \right|^2$

(b) $\| f \|^2 \geq \sum_{j=1}^{\infty} \left| \langle f, e_j \rangle \right|^2$

(c) $\| f \|^2 = \sum_{j=1}^{\infty} \left| \langle f, e_j \rangle \right|^2$

(d) None of the above

3. Give an expression of $t_q(\xi)$, for $q \in \mathbf{Z}$ and $\xi \in \mathbf{R}$.

4. Define MSF wavelet.

5. If $\{k_l \mid l \in \mathbf{Z}\}$ and $\{I_l : l \in \mathbf{Z}\}$ are partitions of K and I respectively, then :

(a) $K_l = 2^{j_l} I_2 j_l$

(b) $K_l = 2^{j_l} I_l$

(c) $K_{2^{j_l}} = I_2 j_l$

(d) None of the above

6. Let $\{v_j \mid j \in \mathbf{Z}\}$ and $\{w_j \mid j \in \mathbf{Z}\}$ closed subspaces of $L^2(\mathbf{R})$, then which of the following is true ?

(a) $V_j = \bigoplus_{l=-\infty}^{j-1} W_l$

(b) $V_j = \bigoplus_{l=1}^{\infty} W_l$

(c) $V_j = \bigcup_{l=1}^{\infty} W_l$

(d) None of the above

7. If ψ is an MRA wavelet, then :

(a) $D_\psi(\xi) < 0 \, \xi \in \mathbf{T}$

(b) $D_\psi(\xi) = 1 \, \xi \in \mathbf{T}$

(c) $D_\psi(\xi) > 0 \, \xi \in \mathbf{T}$

(d) Both (b) and (c) hold

8. Limit of a sequence of MRA wavelets is always a/an :

   (a) Orthonormal wavelet

   (b) Haar wavelet

   (c) MRA wavelet

   (d) MSF wavelet

9. Define tight frame.

10. Define frame operator.

11. Define Zak transform.

12. Define smooth frame for H^2 (**R**).

13. Define Γ_N the subgroup of **T**.

14. Fast Fourier transform is a/an :

   (a) Fourier transform

   (b) Advanced transform

   (c) Algorithm

   (d) Cosine transform

15. Define Dual frame.

16. If A and B are frame bounds of a frame $\left\{g_{m,n} \mid m, n \in \mathbf{Z}\right\}$, then bounds of $\left|\mathrm{Rg}\left(s,t\right)\right|^2$ are .... and ....

17. The orthogonal projection operator $P_j, j \in \mathbf{Z}$ from $L^2(\mathbf{R})$ onto V_j is given by :

(a) $\sum_{k \in \mathbf{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}(x)$

(b) $\sum_{k \in \mathbf{Z}} \langle f, \phi_{j,k} \rangle \phi_{j,k}(x)$

(c) $\sum_{k \in \mathbf{Z}} \langle \phi_{j,k} \psi_{j,k} \rangle \psi_{j,k}(x)$

(d) None of the above

18. The domain and codomain of a frame operator F are .... and .... respectively.

19. If $f \in l^2(\mathbf{Z})$, then $\| f \|_{l^2(\mathbf{Z})}$ = ...

20. Write an expression for Maxian hat function.

Section—B 2 each

(Very Short Answer Type Questions)

Note : Attempt all questions.

1. Give an example of a dense subset of $L^2(\mathbf{R})$.
2. Give the necessary and sufficient conditions for a function $\psi \in L^2(\mathbf{R})$ to be orthonormal wavelet.
3. Write all the characteristics for a function m_0 to be low pass filter.

4. For a 2π-periodic function g on **R** and a set K (to be the set required in the characterization of low pass filter) prove that :

$$\int_{K} g(\xi)\, d_{\xi} == \int_{-\pi}^{\pi} g(\xi)\, d_{\xi}$$

5. Find A and B for $f \in C^2$ and $\{\phi_1, \phi_2, \phi_3\}$ to be tight frame.

6. If A and B are the bounds of $f \in H$, then write frame bounds of $S^{-1} = (F^* F)^{-1}$.

7. If P, Q and T are $n \times n$ matrix, and A # and B^d have usual meanings, prove that :

$$P^{\#}Q^{d} = T^{\#}$$

8. Give statement for fast Fourier transform.

Section—C 3 each

(Short Answer Type Questions)

Note : Attempt any *eight* questions.

1. Suppose that $\{e_j : j = 1, 2, ...\}$ is a system of vectors in a Hilber space **H** satisfying $\|f\|^2 = \sum_{j=1}^{\infty} \left|\langle f, e_j \rangle\right|^2$ for all $f \in \mathbf{H}$. If $\|e_j\| \geq 1$ for $j = 1, 2, ...,$ then prove that $\{e_j : j = 1, 2, ...\}$ is an orthonormal basis for **H**.

2. if ψ is an orthonormal wavelet and $|\hat{\psi}|$ is continuous at zero, then prove that $\hat{\psi}(0) = 0$.

3. Let $\mu_1, \ldots\ldots\ldots\ldots, \mu_n$ be 2π-periodic functions and set :

$$M_j = \sup_{\xi \in \mathbf{T}} \left(\left|\mu_j(\xi)\right|^2 + \left|\mu_j(\xi + \pi)\right|^2 \right)$$

Then prove that :

$$\int_{-2^n\pi}^{2^n\pi} \prod_{j=1}^{n} \left|\mu_j\left(2^{-j}\xi\right)\right|^2 d\xi \leq 2\pi M_1 M_2 \ldots\ldots\ldots\ldots M_n$$

4. Suppose that :

$$\left\{\psi^{(n)} : n = 1, 2, \ldots\ldots\ldots\right\}$$

is a sequence of MRA = wavelet converging to ψ in $L^2(\mathbf{R})$. If ψ is also a wavelet, then prove that ψ is an MRA wavelet.

5. Prove that the translation operator T_k commutes with $\mathbf{F}^*\mathbf{F}$, where **F** is a frame operator.

6. Suppose $g \in L^2(\mathbf{R})$ and $\left\{g_{m,n} : m, n \in \mathbf{Z}\right\}$ is a frame with frame bounds A and B with $0 < A \leq B < \infty$, then show that :

$$0 < A \leq \left|Rg(s,t)\right|^2 \leq B < \infty$$

7. Prove that the vectors :

$$\left\{ \sqrt{\frac{2}{N}} C_k^{(N)} : K = 0, 1, \ldots\ldots\ldots\ldots\ldots N-1 \right\}$$

form an orthonormal system for $\mathbf{R}^N$.

8. Write in short the decomposition algorithm for wavelets.

9. Write the reconstruction algorithms for wavelets.

Section—D

5 each

(Long Answer Type Questions)

Note : Attempt all questions.

1. Let H be a Hilbert space and $\{e_j : j = 1, 2, \ldots\ldots\ldots\}$ be a family of elements of **H**. Then prove that :

(i) $\| f \|^2 = \sum_{j=1}^{\infty} \left| \langle f, e_j \rangle \right|^2$ for all $f \in \mathbf{H}$ if an only if

(ii) $f = \sum_{j=1}^{\infty} \langle f, e_j \rangle e_j$ with convergence in H, for all $f \in \mathbf{H}$.

Or

Let $\{v_j : j \geq 1\}$ be a family of vectors in a Hilbert space H such that :

(i) $\sum_{n=1}^{\infty} \|v_n\|^2 = c < \infty$

(ii) $v_n = \sum_{m=1}^{\infty} \langle v_n, v_n \rangle v_n \quad \forall n \geq 1$

Let F = $\overline{\text{span}\{v_j : j \geq 1\}}$. Then prove that dim F = C.

2. Let $m_0 \in C'(\mathbf{R})$ be 2π-periodic function with $m_0(0) = 1$, $\left|m_0(\xi)^2\right| + \left| m_0(\xi + \pi) \right|^2 = 1$ and there exists a set K $\subseteq$ **R** as a finite union of closed and bounded intervals with O in the interior of K,

$$\sum_{k \in \mathbf{Z}} \chi_k(\xi + 2k\pi) = 1 \text{ for } \xi \in \mathbf{R}$$

and $m_0(2^{-j}\xi) \neq 0$, for $j = 1, 2, ..., \xi \in k$. Then prove that m_0 is a law pass filter for an MRA.

Or

A function $\phi \in L^2(\mathbf{R})$ is a scaling function for an MRA if and only if :

(i) $\sum_{k \in \mathbf{Z}} \left|\hat{\phi}(\xi + 2k\pi)\right|^2 = 1 \ \xi \in \mathbf{T}$

(ii) $\lim_{j-\infty} \left|\hat{\phi}\left(2^{-j}\xi\right)\right| = 1 \quad \xi \in \mathbf{R}$

(iii) There exists a 2π-periodic function m_0 such that :

$$\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi) \quad \xi \in \mathbf{R}$$

3. State and prove Bolian low theorem for frames.

Or

Suppose that $\{e_j : j = 1, 2, ...\}$ is a family of elements in a Hilbert space H such that $0 < A \le B < \infty$ satisfying :

$$A\|f\|^2 \le \sum_{j=1}^{\infty} \left|\left\langle f, e_j \right\rangle\right|^2 \le B\|f\|^2$$

for all f beloriging to dense subset D of **H**. Then prove that above inequality holds for all $f \in \mathbf{H}$.

4. Write in details about discrete cosine transform and fast cosine transform.

Or

If $N = 2^q$, then prove that $C_N = E_1 E_2 .... E_q$, where each E_j is an $N \times N$ matrix such that each row has precisely two non-zero entries.