F - 520

M.A./M.Sc.(Second Semester)
EXAMINATION, May - June, 2022
MATHEMATICS
Paper Second
(Real Analysis - II)

Time : Three Hours]   *[Maximum Marks:80*

Note: Attempt all the sections as directed.

(Section - A)

(Objective/Multiple Choice Questions)

(1 mark each)

Note : Attempt all the questions.

    Choose the correct answer.

1. Let f be a bounded function and α a monotonically increasing function on [a, b] then .

   (A) $\displaystyle \int_{\underline{a}}^{b} f\,d\alpha = \int_{a}^{\overline{b}} f\,d\alpha$

   (B) $\displaystyle \int_{\underline{a}}^{b} f\,d\alpha \le \int_{a}^{\overline{b}} f\,d\alpha$

   (C) $\displaystyle \int_{\underline{a}}^{b} f\,d\alpha \ge \int_{a}^{\overline{b}} f\,d\alpha$

   (D) None of the above

2. Statement - I Let f be continuous and be α monotonically increasing on [a, b] then $f \in R(\alpha)\ on\ [a,b]$.

   Statement - II Let f be monotonic on [a, b] then $f \in R(\alpha)$

   (A) Only statement I is true
   (B) Only statement II is true
   (C) Both statement I & II is true
   (D) Both statement I & II is false

3. Let $f(x) = x, \alpha(x) = x^2$. The value of $RS\displaystyle\int_{0}^{1} x\,d\,x^2$ is

   (A) $\dfrac{1}{2}$

   (B) $\dfrac{2}{3}$

   (C) 1

   (D) 0

4. Let *f* be a Riemann Integral on [a, b] is $f \in R\ [a,b]$ and let there be differentiable function F on [a, b] such that $F' = f$

then $\int_{a}^{b} f(x)\,dx = F(b) - F(a)$

   (A)   Fundamental theorem of calculus

   (B)   Intergration thoerem

   (C)   Differentiation theorem

   (D)   None of the above

5. If E is Q set for which m is defined and if *E + y* is the set $\{x+y : x \in E\}$ where *y* is any fixed number then $m(E+y) = m(E)$. This property is known as:

   (A)   Rotational invarience

   (B)   Translation invarience

   (C)   Countable addivity

   (D)   None of the above

6. Which of the following is true

   (A)   The outer measure of an interval is its length

   (B)   The outer measure of an interval is its breath.

   (C)   The outer measure of an interval is either length or breath

   (D)   The outer measure of an interval is neither length nor breath.

7. A Borel measurable set is _____ measurable is.

   (A)   Lebesgue measurable

   (B)   Borel Measurable

   (C)   Regular measurable

   (D)   None of the above

8. Any set with the outer measure different from zero is:

   (A)   Countable

   (B)   Uncountable

   (C)   Measurable

   (D)   Non - Measurable

9. Let ϕ and Ψ be simple function which vanish outside a set of finite measure then:

   (A) $\int (a\,\phi + b\,\Psi) = a\int \phi + b\int \Psi\ \forall\, a,\, b \in R$

   (B) $\int (a\,\phi + b\,\Psi) \le a\int \phi + b\int \Psi\ \forall\, a,\, b \in R$

   (C) $\int (a\,\phi + b\,\Psi) \ge a\int \phi + b\int \Psi\ \forall\, a,\, b \in R$

   (D) $\int (a\,\phi + b\,\Psi) \ne a\int \phi + b\int \Psi\ \forall\, a,\, b \in R$

10. A simple function is _____

   (A)   Characteristic function

   (B)   Measurable function

   (C)   Borel function

   (D)   None of them

11. Let f be an integrable function on [a, b] if

$$\int_{a}^{x} f(t)\, dt = 0 \; for\; all\; x \in [a,\, b] \text{ then}$$

(A) $f' = 0 \; a.e \; in \; [a,b]$

(B) $f' \neq 0 \; a.e \; in \; [a,b]$

(C) $f = 0 \; a.e \; in \; [a,b]$

(D) $f \neq 0 \; a.e \; in \; [a,b]$

12. If a function f is a function of bounded variation then it is:

(A) Bounded

(B) Measurable

(C) Unbounded

(D) None of the above

13. Every obsolutely continuous function is of:

(A) Bounded variation

(B) Continuous variation

(C) Countably infinite

(D) None of the above

14. If the derivatives of two absolutely continuous functions are equivalent then the function differ by a

(A) Derivative

(B) Constant

(C) Integrable

(D) Closed

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P.T.O.

15. If f is an absolutely continuous monotone function on [a, b] and E a set of measure zero then

(A) f(E) has an infinite measure

(B) f(E) has finite measure

(C) f(E) has measure zero

(D) None of the above

16. If F'(x) be an indefinite integral of a bounded measurable function f(x) then

(A) $F'(x) < f(x)$ almost every where

(B) $F'(x) > f(x)$ almost every where

(C) $F'(x) = f(x)$ almost every where

(D) $F'(x) \neq f(x)$ almost every where

17. If $f \in L^{P}\,[a,b]$ and $g \leq f$ then

(A) $g \in L^{P}\,[a,b]$

(B) $g \notin L^{P}\,[a,b]$

(C) $(f + g) \in L^{P}\,[a,b]$

(D) $(f + g) \notin L^{P}\,[a,b]$

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18. Let f be a real valued and measuable function on a set X with $\mu(x) > 0$. A real number M is said to an essential bounded for the function f if

   (A) $|f(x)| \leq M$ a.e on X

   (B) $|f(x)| \geq M$ a.e on X

   (C) $|f(x)| = M$ a.e on X

   (D) $|f(x)| \neq M$ a.e on X

19. Let $\{f_n\}$ be a sequence of measurable function that converges in measure to f for some $f \in R$ then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges to f a.e on measure space X.

   (A)   Lebesgue theorem
   (B)   Convergence theorem
   (C)   Riesz theorem
   (D)   None of the above

20. Let $\{f_n\}$ be a sequence of function in L^P, $1 \leq p < \infty$ which converges almost every where to a function f in L^P then $\{f_n\}$ converges to f in L^P if and only if

   (A) $\|f_n\|_p \to \|f\|_p$

   (B) $\|f_n\|_p \nrightarrow \|f\|_p$

   (C) $\|f_n\|_p\, 0$

   (D)   None of the above

Section - B

(Very Short Answer Type Questions)

(1½ marks each)

Note: Attempt all questions.

1.   State fundamental theorem of calculus.

2.   Define Rectifiable curve

3.   Define Borel Measurable set

4.   Give two examples of Borel set

5.   State Fatou's Lemma.

6.   State Lebesgue dominated convergence theorem

7.   Define Integrable function.

8.   Define Absolutely continuous function

9.   Give an example which is continuous but not absolutely continuous.

10. Define a function of Bounded Variation

Section - C

(Short Answer Type Questions)

(2½ marks each)

Note: Attempt all questions.

1.   Let f be continuous and α be monotonically increasing on [a, b] then $f \in R(\alpha)$ on [a, b]

2. Let $f, \alpha = [a,b] \to R$ be bounded function and α be monotonic increasing. If P is any partition of interval [a, b] then $L(P, f, \alpha) \leq U(P, f, \alpha)$.

3. Prove that the union of a finite number of measurable set is measurable.

4. Prove that a continuous function define on a measurable set is measurable.

5. Prove that if f is a function of bounded variation on [a, b] then it is measurable.

6. Prove that if f is absolutely continuous on [a, b] and $f' = 0 \, a.e$ then f is constant.

7. Prove that the intersection of two outer measurable set is outer measurable.

8. If $f \in L^P[a, b], p > 1$ and $f \in L[a,b]$.

9. If $A \in A$ where A is set of Algebra then prove that $\mu^*(A) = \mu(A)$

10. Let $f \in L(\mu)$ on E then $|f| \in L(\mu)$ on E and
$$\left| \int_E f \, d\mu \right| \leq \int_E |f| \, d\mu$$

Section - D

(Long Answer Type Questions)

(4 marks each)

Note: Attempt all questions.

1. Let $f \in R$ on $[a,b]$ for $a \leq x \leq b$ put $F(x) = \int_a^x f(t) dt$.

   Then prove that F is continuous on [a, b]. Further more if f is continuous at a point x_0 of [a, b] then F is differentiable at x_0 and $F'(x_0) = f(x_0)$

   OR

   Let α be a monotonically increasing function on [a, b] and $\alpha' \in R[a,b]$. Let F be a bounded real function on [a, b] then $f \in R(\alpha)$ if and only if $f\alpha' \in R[a,b]$. In that case
   $$\int_a^b f d\alpha = \int_a^b f(x) \, \alpha'(x) \, dx$$

2. State and prove Jordan decomposition theorem

   OR

   Let $\{f_n\}$ be a sequence of non-negetive measurable function and $f_n \to f$ a.e on E then prove that $\int_E f \leq \lim_{n \to \infty} \int_E f_n$

3. Let f be a bounded function defined on [a, b] if f is Riemann integrable on [a, b] then it is Lebesgue inte-grable on [a, b] and $R.\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$.

OR

Let the set E_1, E_2 ,...........,E_n be disjoint measurable then prove that $\mu*\left[A\cap(\bigcup_{i=1}^{n} E_1)\right] = \sum_{i=1}^{n}\mu*(A_n, E_i)$ holds for every subset A of X.

4. State and prove Lebesgue differentation theorem.

OR

Prove that a function f is of bounded variation on [a, b] if and only if f is the difference of two monotone real valued function on [a, b].

5. State and prove Minkowski's inequality.

OR

Prove that L^P spaces are complete.