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M.A/M.Sc. (Fourth Semester)
EXAMINATION, MAY-JUNE, 2022
MATHMATICS
Paper First
(Functional Analysis–II)

Time : Three Hours] *[Maximum Marks : 80*

Note : Alltempt all questions.

(Section-A)

(Objective/Multiple Choice Questions)

(1 mark each)

Note- Attempt all questions.

Choose the most appropriate answer.

1. Let X be an arbitary normed linear space the mapping f : X $\rightarrow$ X** is an Isometrically isomorphic from X into X** if
   (A) It is linear
   (B) It is bounded
   (C) It preserves distance
   (D) All the above

2. If S is a subspace of normed linear space X then
   (A) $S = S^1$
   (B) $\overline{S} = S_0$
   (C) $\overline{S} = S_1^1$
   (D) $S_0 = S_1^1$

3. Let T be a closed linear map of a Banach space X into a Banach space Y then T is
   (A) Isomorphic
   (B) Continuous
   (C) Uniformly Continous
   (D) Homeomorphic

4. A normed linear space X is said to be ________ if J is onto i.e. J (X) =X**
   (A) Isomorphic
   (B) Adjoint
   (C) Reflexive
   (D) Symmetric

5. Let X be a normed linear space over field K and let M be a linear sub space of X. Suppose that x_0 be a vector not is M and $d(x_0, M) = d > 0$ then there exist $g \in X^*$ such that -

(A) $g(M) = \{0\},\ g(x_0) = d \ \ and\ \ \|g\| = 1$

(B) $g(M) \neq \{0\}, g(x_0) = d\ and\ \|g\| = 1$

(C) $g(M) = \{0\},\ g(x_0) > d \ \ and\ \|g\| \neq 1$

(D) $g(M) = \{0\},\ g(x_0) \neq d \ \ and\ \|g\| \neq 1$

6. The set of all compact linear operators from X into Y forms a :

(A) Real Space

(B) Vector Space

(C) Complex Space

(D) None of the above

7. Let T be a bounded operator in a Hilbert space H. If λ is an Eigen value of T then

(A) $|\lambda| \leq \|T\|$

(B) $|\lambda| \geq \|T\|$

(C) $|\lambda| = \|T\|$

(D) $|\lambda| \neq \|T\|$

8. Every bounded linear functional defined on subspace of real normed space may be extended linearly with pre-sentation of the norm to the whole of x :

(A) Banach theorem

(B) Hahn Banach theorem

(C) Banach Steinhaus theorem

(D) Projection theorem

9. In an inner product space, the inner product is -

(A) Uniformly continuous

(B) Jointly continuous

(C) Absolutely continuous

(D) Continuous

10. A Hilbert space H is ________ if it has a countable orthonormal basis

(A) Reflexive

(B) Compact

(C) Seperable

(D) Isometrically Isomorphic

11. If X and Y are any two vectors in an inner product space X then

$$|< x, y >| \leq \|x\| \, . \, \|y\|$$

The above inequality is known as

(A) Parallelogram Law

(B) Polarisation Identity

(C) Cauchy-Schwarz Inequality

(D) Bessel's Inequality

12. Let H be a Hilbert space and let {e i} be an ortho normal set in H then

$$\|x^2\| = \sum |< x, ei >|^2$$

This inqequality is known as

(A) Parseval's identity

(B) Holder's inequality

(C) Minkowski's inequality

(D) None of these

13. Statement (i) Every self-adjoint operator is Normal

Statement (ii) Every Normal vector is unitary

(A) Only (i) is correct

(B) Only (ii) is correct

(C) Both (i) & (ii) Correct

(D) Both (i) & (ii) are incorrect

14. A linear operator (T, D (T)) is said to be bounded if -

(A) $sup \{\|Tx\| : x \in D\ (T),\ \|x\| \leq 1\} = \infty$

(B) $Inf\ \{\|Tx\| : x \in D\ (T),\ \|x\| \leq 1\} = \infty$

(C) $Sup\ \{\|Tx\| : x \in D\ (T),\ \|x\| \leq 1\} \leq \infty$

(D) $Inf\ \{\|Tx\| : x \in D\ (T),\ \|x\| \leq 1\} < \infty$

15. An operator T is called unitary of

(A) T = T*

(B) T* T = T T * = I

(C) T* T = T T*

(D) None of the above

16. Let H be a Hilbert space and let the mapping $\psi : H \rightarrow H^2$ be defined by $\psi\ (y) = f_y\ ,\ f_y(x) = < x, y > \forall x, y \in H$ then :

(A) ψ is one-one and onto

(B) ψ is not linear

(C) ψ is ometry

(D) All the above

17. Let M be a linear subspace of a Hilbert space H then M is closed if and only if -

(A) $M = M^1$

(B) $M = M^{11}$

(C) $M^1 = M^{11}$

(D) $M \neq M^1$

18. Let H be a seperable infinite dimensional complex Hilbert space then H is isometrically isomorphic to l_2

(A) Riesz Representation theorem

(B) Fourier theorem

(C) Riesz-Fischer theorem

(D) None of the above

19. Let T be a bounded self adjoint operator on a Hilbert space H then

(A) $\|T\| = Sup\{|<x, Tx>| : \|x\| = 1\}$

(B) $\|T\| = Sup\{|<x, Tx>| : \|x\| \neq 1\}$

(C) $\|T\| = Inf\{|<x, Tx>| : \|x\| = 1\}$

(D) $\|T\| = Inf\{|<x, Tx>| : \|x\| \neq 1\}$

20. Let X and Y be two Banach spaces an ordered pair (T, D(T)) when D(T) is a linear subspace in X and T is a linear map from D(T) into y is called a ________ operator from X into Y -

(A) Bounded Operator

(B) Linear Operator

(C) Closed Operator

(D) Absolutely continuous

(Section- B)

(Very Short Answer Type Questions)

(2 marks each)

Note : Attempt all questions. Answer in 2-3 sentences.

1. State uniform Bounded principle.

2. Define open Map, Closed map and Graph of linear transformation.

3. Define Inner product space.

4. State Riesz Representation theorem.

5. Prove that every positive operator is self-Adjoint.

6. Let T be an operator on H and $T \to T^*$ is a mapping of B(H) into itself for T_1 and $T_2 \in B(H)$ prove that

$$(T_1 + T_2)^* = T_1^* + T_2^*$$

7. Derive Parallelogram law

8. State Bessel's Inequality.

(Section - C)

(Short Answer Type Questions)

(3 marks each)

Note : Attempt all questions. Answer in 75 words.

1. Let N and N' be normed linear space and DCN. Prove that a linear transformation $T:D \rightarrow N'$ is closed if and only if G_T is closed.

2. State and prove Cauchy Schwarz Inequality.

3. Prove that every inner product space is normed space.

4. Let X and Y be Banach space and $T \in B(X,Y)$ If T is onto then there exist $K > 0$ such that for every $y \in Y$ there exist $x \in X$ such that

   $$Tx = y\|x\| \leq K\|y\|$$

5. Prove that a closed subspace of reflexive Banach space is reflexive.

6. Let T be a bounded linear operator on a Hilbert space H then prove that T is normal $\|T * x\| = \|Tx\| \forall x \in H$

7. Show that the product of two bounded self adjoint operator S and T on a Hilbert space H is self adjoint if and only if the operators commute

8. State and prove Parseval's Identity.

Section D

(Long Answer Type Questions)

(5 marks each)

Note:- Attempt all questions. Answer in 150 words.

1. State and prove open mapping theorem.

OR

Prove that a closed linear map T mapping a normed linear space X of the second category into a Banach space Y is continuous.

2. State and prove that Hahn Banach Theorem.

OR

Prove that every Hilbert space is reflexive.

3. Let C be a non-empty closed and convex set in a Hilbert space H then there exist a unique vector in C of smallest norm.

OR

State and prove Closed Rage theorem for Banach space.

4. Let y be a fixed vector in a Hilbert space H and let f_y be a scalar valued function on H defined by

$$f_y(x) = \langle x, y \rangle \ \forall x \in H$$

Then f_y is a functional in H* ie f_y is a continuous linear functional on H and $\|y\| = \|fy\|$.

OR

State and prove Generalized Lax-Milgram theorem.